Laplace-Pólya integrals: the crossroad of analysis, geometry, probability, and combinatorics

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joint work with Barnabás Gárgyán

Discrete Geometry Days[3]

2 July 2024

Central hyperplane sections of the cube

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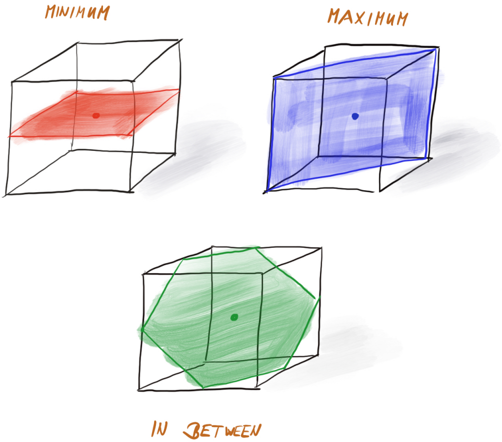
Question: for which $\mathbf{v} \in S^{n-1}$ is $V_n(\mathbf{v})$ minimal or maximal?

Central sections in 3 dimensions



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- Also proved by Vaaler (1979) – for lower dimensional sections as well: minimal k-dimensional sections are parallel to k-dimensional faces

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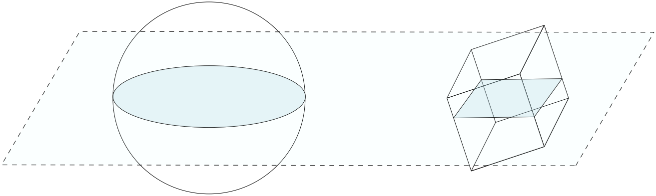
(The cube is surprisingly fat in every direction.)

Counterexample to the Busemann-Petty conjecture

Characterization of maximal central sections also implies that the cube-ball pair is a counterexample to the Busemann-Petty conjecture when $n \geq 10$.



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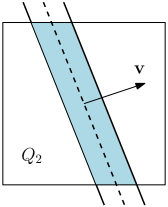
- Pólya (1913): formula for general central hyperplane sections: for any unit vector $\mathbf{v} \in S^{n-1}$,

$$V_n(\mathbf{v}) = \frac{1}{\pi} \int_{-\infty}^{\infty} \prod_{i=1}^{n} \operatorname{sinc} v_i t \, \mathrm{d}t,$$

where

$$\operatorname{sinc} x := \begin{cases} \dfrac{\sin x}{x} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

Probabilistic approach



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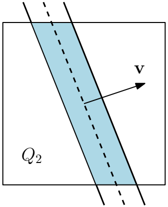

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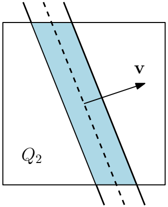

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$$4\varepsilon \cdot f_{\sum_{i=1}^n v_i X_i}(0) \approx \mathbb{P}\left(\left| \frac{1}{2} \sum_{i=1}^n v_i X_i \right| \leq \varepsilon \right)$$

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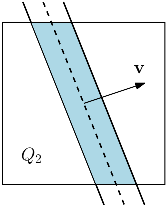

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- Inversion formula:

$$f_X(r) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itr} \varphi_X(t) \, \mathrm{d}t.$$

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$$V_n(\mathbf{v}) = \frac{1}{\pi} \int_{-\infty}^{\infty} \prod_{i=1}^n \operatorname{sinc} v_i t \, \mathrm{d}t.$$

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- Hensley: (1979) Central Limit Theorem implies that

$$\lim_{k\to\infty} \nu_k = \sqrt{\frac{6}{\pi}} \approx 1.3819$$

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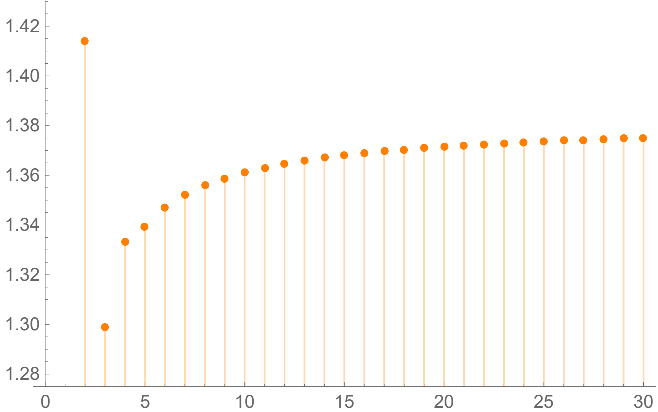

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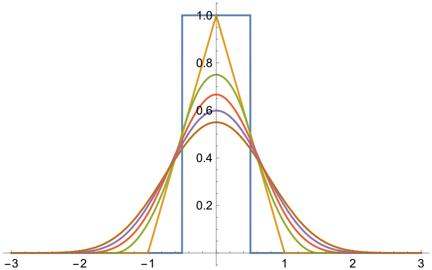
Distribution of $\sum_{i=1}^n X_i$: Irwin-Hall distribution



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Question: are all critical sections diagonal?

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(volume of off-diagonal sections)

Formulae for $J_n(r)$

Asymptotic formula:

$$J_n(0) = \sqrt{\frac{6}{\pi n}}\left(1 - \frac{3}{20n} - \frac{13}{1120n^2} + \frac{27}{3200n^3} + O\left(\frac{1}{n^4}\right)\right).$$

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Exact calculation:

$$J_n(r) = \frac{1}{2^{n-1}(n-1)!} \sum_{i=1}^{\lfloor \frac{n+r}{2} \rfloor} (-1)^i \binom{n}{i} (n+r-2i)^{n-1}$$

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Recursion: Thompson (1966)

$$J_n(r) = \frac{n+r}{2(n-1)}J_{n-1}(r+1) + \frac{n-r}{2(n-1)}J_{n-1}(r-1)$$

Eulerian numbers

Eulerian numbers of the first kind: $A(m, l)$ is the number of such permutations of $\{1, \ldots, m\}$ for which there are exactly l elements which are greater than the previous element (i.e. the permutation has l ascents)

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Thus, if $n + r$ is even,

$$J_n(r) = \frac{1}{(n-1)!} A\left(n-1, \frac{n+r}{2}\right).$$

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- Critical sections: in order to prove that there exist critical sections which are non-diagonal, it suffices to show that

$$\frac{J_{n+2}(0)}{J_n(0)} < \frac{n+2}{n+3}$$

Rate of decrease

Lesieur and Nicolas (1992) proved fine estimates for Eulerian numbers using asymptotic expansion and technically involved calculations: if n is even, then

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Theorem (A., Gárgyán '24+)

For every $n \geq 3$,

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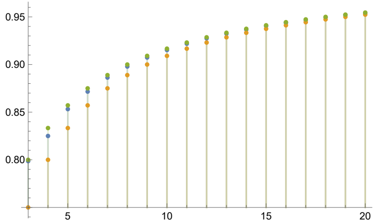

Transforming to horizontal estimates

Due to the recursive equation property

$$J_n(r) = \frac{n+r}{2(n-1)} J_{n-1}(r+1) + \frac{n-r}{2(n-1)} J_{n-1}(r-1)$$

one can show that

$$\frac{J_{n+2}(0)}{J_n(0)} = \frac{(n+2)^2}{2n(n+1)} \cdot \frac{J_n(2)}{J_n(0)} + \frac{n+2}{2(n+1)}$$

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For $r = -1$: $\frac{J_n(-1)}{J_n(1)} = 1$;for $r = n-2$: $\frac{J_n(n)}{J_n(n-2)} = 0.$

Estimates on the rate of decay of the Laplace-Pólya integral

Theorem (A., Gárgyán '24+)

Let $n \geq 4$ *and* r *be integers satisfying* $-1 \leq r \leq n-2$. *Then*

$$c_{n,r} \leq \frac{J_n(r+2)}{J_n(r)} \leq d_{n,r},$$

where

$$c_{n,r} = \frac{(n-r)^2}{(n+r+2)^2} \cdot \frac{(4n-7r-8)(4n+3r+6)}{(4n+7r+6)(4n-3r)}$$

and

$$d_{n,r} = \frac{(n-r)}{(n+r+2)} \cdot \frac{(n-r)^2-2}{(n+r+2)^2}$$

The proofs are entirely combinatorial, based on the recursive formula

$$J_n(r) = \frac{n+r}{2(n-1)} J_{n-1}(r+1) + \frac{n-r}{2(n-1)} J_{n-1}(r-1).$$

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Corollary

For every $m \geq 2l-1$ and $2 \leq l \leq m$,

$$A(m, l-1) < \frac{l^3}{(m-l+1)^3} A(m, l).$$

Existence of non-diagonal critical sections of the cube

Using the characterization of critical sections (Ambrus, Proc. AMS '22):

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Using the characterization of critical sections (Ambrus, Proc. AMS '22):

Ambrus, Gárgyán, Adv. Math '24:

Theorem

For every $n \geq 4$ there exists a non-diagonal critical central section of Q_n whose normal vector is of the form $\mathbf{v} = (a, a, b, \ldots, b) \in \mathbb{R}^n$ with $a, b > 0$, $a \neq b$.

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- The found non-diagonal critical point is a saddle point – Question: are all locally extremal sections diagonal?

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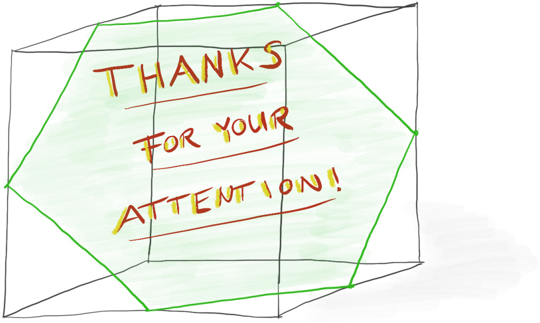
- Concavity and log-concavity of central diagonal sections
- Lower dimensional sections, other convex bodies, projections, etc. etc.


THANKS
FOR YOUR
ATTENTION!