Subspace Concentration Conditions and Cone-Volumes

A Discrete Approach

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The Discrete Logarithmic Minkowski Problem

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- $vol(C_i) = \frac{b_i}{n} vol_{n-1}(F_i)$.
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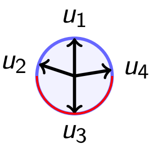

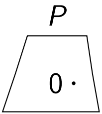

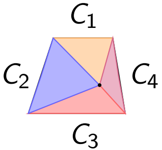

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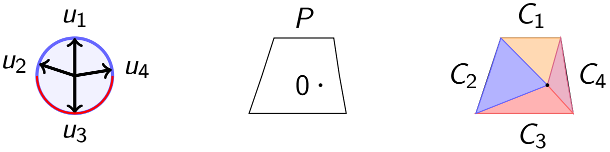

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Characterize the cone-volume measure $V_P(\cdot)$ of a polytope P.

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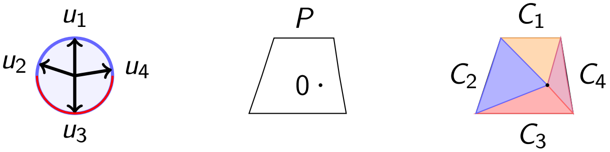

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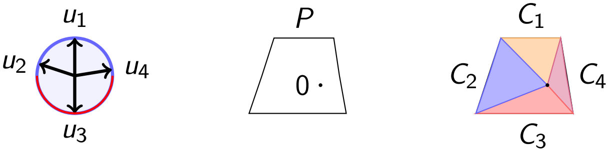

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Characterize the cone-volume measure $V_P(\cdot)$ of a polytope P.

- The symmetric case is fully understood:
- K is called symmetric if $K = -K$.
- A measure μ is called *even* if $\mu(\omega) = \mu(-\omega)$ for all $\omega \subset \mathbb{S}^{n-1}$.

Symmetric Case

Theorem Börözky, LYZ, 2013:

An even finite Boreal measure μ is the cone-volume measure $V_P(\cdot)$ of an $n-$dimensional symmetric polytope P if and only if μ satisfies the *subspace concentration condition* (scc):

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Necessity was independently shown by B. He, G. Leng, K. Li, 2006 and H., Schürmann, Wills, 2005.

SCC for Polytopes

The scc for P reads:

i) for every proper linear subspace $L \subset \mathbb{R}^n$ it holds

$$\frac{V_P(L \cap \mathbb{S}^{n-1})}{V_P(\mathbb{S}^{n-1})} = \frac{1}{vol(P)} \sum_{u_i \in L} vol(C_i) \leq \frac{\dim(L)}{n},$$

ii) equality holds in i) for a subspace L if and only if there exists a complementary subspace L' and polytopes $Q_1 \subset L$ and $Q_2 \subset L'$ such that $P = Q_1 \oplus Q_2$.

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$$L(U) := \{S \subset U : 1 \leq rk(S) \leq n-1 \text{ and } U \cap \text{lin}(S) = S\},$$
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We define the *subspace concentration polytope*:

$$P_{scc}(U) := \Big\{x \in \mathbb{R}^m : x_i \geq 0, \sum_{i=1}^{m} x_i = 1,$$
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Y.-D. Liu, Q. Sun, G. Xiong, 2024: called concentration polytope for U without antipodal points.

Basis Matroid Polytope

Observation:

$\gamma = (\gamma_1, \ldots, \gamma_m) \in P_{scc}(U)$ if and only if $\mu := \sum_{i=1}^{m} \gamma_i \delta_{u_i}$ satisfies the scc and $\mu(\mathbb{S}^{n-1}) = 1$.

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Basis Matroid Polytope Feichtner, Sturmfels, 2005:

$$P(U) := \Big\{ x \in \mathbb{R}^m : x_i \geq 0, \sum_{i=1}^m x_i = n, \sum_{i: u_i \in S} x_i \leq rk(S), \text{ for all } S \in L(U) \Big\}$$
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From the definition it follows $n \cdot \overline{P_{scc}(U)} = P(U)$.

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$$P_{scc}(U) = \frac{rk(V)}{n} P_{scc}(V) \times \frac{rk(U \setminus V)}{n} P_{scc}(U \setminus V).$$

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- It is enough to only consider irreducible U.

Cone-Volume Set

- For $b \in \mathbb{R}^m_{\geq 0}$ let

$$P_U(b) = \{x \in \mathbb{R}^n : \langle u_i, x \rangle \leq b_i, 1 \leq i \leq m\},$$

and let

$$\gamma(b) := \left(vol_{n-1}(F_1)\frac{b_1}{n}, \dots, vol_{n-1}(F_m)\frac{b_m}{n} \right)$$

be its *cone-volume vector*. Here $F_i = P_U(b) \cap \{\langle u_i, x \rangle = b_i\}$ and might be empty or of dimension less than $n - 1$.

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- Sh. Chen, Q.-R. Li, G. Zhu, 2019: $P_{scc}(U) \subset C_\gamma(U)$.

Cone-Volume Set and Separators

The cone-volume set can be decomposed as well:

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This decomposition implies

$$\text{aff}(P_{scc}(U)) = \text{aff}(C_\gamma(U)).$$

Further,

$$\dim(P_{scc}) = \dim(C_\gamma(U)) = |U| - |\{S \in F(U) : S \text{ is irreducible}\}|.$$

Examples

- Let $U = (u_1, \ldots, u_{n+1}) \subset \mathbb{R}^{n \times n+1}$ be in general prosition. Then

$$\overline{P_{scc}(U)} = conv\left(\left\{\frac{1}{n}\sum_{i=1, i\neq j}^{n+1} e_i : j = 1, \ldots, n+1\right\}\right)$$

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- Pollehn, 2018. Given pairwise distinct $u_1, u_2, u_3 = -u_1, u_4 \in \mathbb{S}^1$, $u_2 \neq -u_4$, positively spanning $\mathbb{R}^2$. Then

$$C_\gamma(U) = \{\gamma \in \mathbb{R}^4_{\geq 0} : |\gamma|_1 = 1, \gamma_1 + \gamma_3 < \gamma_2 + \gamma_4\}$$
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- In general, $C_\gamma(U)$ is not convex (cf. Böröczky, Hegedűs, 2015)

Structure of $C_\gamma(U)$

- Y. Liu, X. Lu , Q. Sun, G. Xiong, 2024. Sufficient and necessary conditions for quadrilaterals.

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- Y. Liu, X. Lu , Q. Sun, G. Xiong, 2024. Sufficient and necessary conditions for quadrilaterals.
- Baumbach, Henk, 2024+. $C_\gamma(U) = P_{scc}(U)$ if and only if U defines a parallelepiped.
  If $U = (\pm e_1, \dots, \pm e_n)$. Then

$$
\begin{aligned}
P_{scc}(U) &= \frac{1}{n}\{x \in \mathbb{R}^2 : x \geq 0, x_1 + x_2 = 1\}^n \\
&= \frac{1}{n} conv \left(\sum_{i=1}^{n} \epsilon_i e_{2i-1} + (1-\epsilon_i) e_{2i} : \epsilon_i \in \{0,1\} \right).
\end{aligned}
$$

$\overline{C_\gamma(U)}$ is semialgebraic

- A set $M \subset \mathbb{R}^n$ is called semialgebraic if we can write

$$M = \bigcup_{i=1}^{k} \{x \in \mathbb{R}^n : f_{i_1}(x), \dots, f_{i_{l_i}}(x) > 0, g_i(x) = 0\},$$

for some polynomials $f_{i_j}, g_i \in \mathbb{R}[x_1, \dots, x_n]$.

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- if $P_U(b)$ is simple of $a-$type $\mathcal{A}$ (strongly isomorphic), then

$$vol(P_U(b)) = \sum a_{j_1 \dots j_n} b_{j_1} \cdots b_{j_n} \in \mathbb{R}[b_1, \dots, b_m].$$

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- Every polytope $P_U(\tilde{b})$ can be approximated by a simple polytope $P_U(b)$.
- Baumbach, Henk, 2024+: $\overline{C_\gamma(U)}$ is semialgebraic.

Planar Case, $n = 2$

Define the following sets:

$$U_\triangle := \{u \in U : \exists v, w \in U \, s.t. \operatorname{pos}\{u, v, w\} = \mathbb{R}^2\},$$
$$U_\square := U \setminus U_\triangle$$
$$U_{\square,u} := \{x, y : \operatorname{pos}\{u, -u, x, y\} = \mathbb{R}^2\} \text{ for } u \in U_\square.$$

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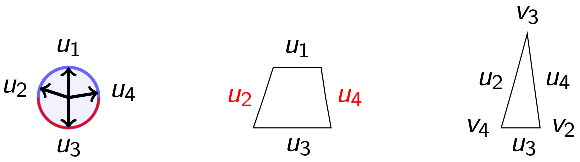

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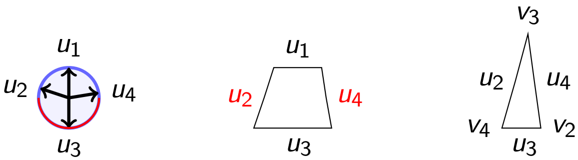

Baumbach,Henk 2024+: We get the $\mathcal{V}-$representation

$$conv(\overline{C_{\gamma}(U)}) = conv\left(\bigcup_{i:u_i \in U_{\triangle}} \{e_i\} \cup \bigcup_{i:u_i \in, u_j \in U_{\square,u_i}} \{\frac{1}{2}(e_i + e_j)\}\right).$$

Table of Contents

Outlook

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$$conv(\overline{C_\gamma(U)})$$
$$=$$
$$conv\left(\left\{\sum_{k=1}^{r} \frac{rk(\Delta_k)}{n} e_{j_k} : u_{j_k} \in (\Delta_k + c_{k-1}), \Delta_1, \ldots, \Delta_r \right.\right.$$
$$\left.\left.\text{is a simplicial partition of } U\right\} \cup \left\{\bigcup_{i=1: u_i \in U_\triangle}^{|U|} e_i\right\}\right).$$

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- $C_{\gamma,sym}(U)$ is convex using the volume formulas for simple polytopes.

Thank you for your attention!