From Non-Separable Arrangements To Separable Packings

Karoly Bezdek **Canada Research Chair**
Univ. of Calgary, Canada **and** Univ. of Pannonia, Hungary

Topics to be discussed:

- Minimal coverings of non-separable arrangements of convex bodies (resp., spherical caps) in Euclidean spaces (resp., spherical spaces)
- Dense totally separable packings of convex bodies (resp., congruent spherical caps)
- Contact numbers for locally separable sphere packings

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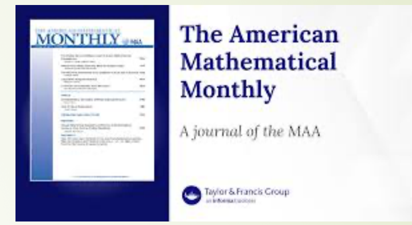
Part I – Minimal Coverings of Non-Separable Arrangements

Pat I/A – Non-separable arrangements in Euclidean spaces



[GG45] A. W. Goodman and R. E. Goodman, *A circle covering theorem*, Amer. Math. Monthly **52** (1945), 494–498. MR13513

2.1. **Non-separable arrangements in Euclidean spaces.** The following theorem was conjectured by Erdős and proved by Goodman and Goodman in [GG45].

Theorem 2.1. *Let the disks* $\mathbf{B}[\mathbf{x}_1, \tau_1] \subset \mathbb{E}^2, \dots, \mathbf{B}[\mathbf{x}_n, \tau_n] \subset \mathbb{E}^2$ *have the following property: No line of* $\mathbb{E}^2$ *divides the disks* $\mathbf{B}[\mathbf{x}_1, \tau_1], \dots, \mathbf{B}[\mathbf{x}_n, \tau_n]$ *into two non-empty families without touching or intersecting at least one disk. Then the disks* $\mathbf{B}[\mathbf{x}_1, \tau_1], \dots, \mathbf{B}[\mathbf{x}_n, \tau_n]$ *can be covered by a disk of radius* $\tau := \sum_{i=1}^{n} \tau_i$.

[Had47] H. Hadwiger, *Nonseparable convex systems*, Amer. Math. Monthly **54** (1947), 583–585.

Recall that a convex domain of $\mathbb{E}^2$ is a compact convex set with non-empty interior. Hadwiger [Had47] extended Theorem 2.1 by introducing the concept of *non-separable system* as follows: A system of convex domains $\mathbf{K}_i \subset \mathbb{E}^2$, $i = 1, \ldots, n$ is called *separable* if there is a line of $\mathbb{E}^2$, which is disjoint from each $\mathbf{K}_i$ and which divides $\mathbb{E}^2$ into two open half planes each containing at least one $\mathbf{K}_i$. In the opposite case we call the system *non-separable*. Projections on lines and Cauchy's perimeter formula combined with Theorem 2.1 yield the following inequalities.

Theorem 2.3. *Let $\mathbf{K}_i$, $i = 1, \ldots, n$ be a non-separable system of convex domains in $\mathbb{E}^2$. If $\mathbf{K}_0 := \operatorname{conv}(\cup_{i=1}^n \mathbf{K}_i)$ and $\operatorname{per}(\mathbf{K}_i)$, $\operatorname{diam}(\mathbf{K}_i)$, and $\operatorname{cr}(\mathbf{K}_i)$ denote the perimeter, the diameter, and the circumradius, respectively, of the convex domain $\mathbf{K}_i$, $i = 0, 1, \ldots, n$, then*

$$(2) \qquad \operatorname{per}(\mathbf{K}_0) \leq \sum_{i=1}^{n} \operatorname{per}(\mathbf{K}_i), \ \operatorname{diam}(\mathbf{K}_0) \leq \sum_{i=1}^{n} \operatorname{diam}(\mathbf{K}_i), \ \text{ and } \operatorname{cr}(\mathbf{K}_0) \leq \sum_{i=1}^{n} \operatorname{cr}(\mathbf{K}_i).$$

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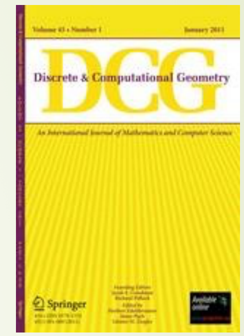
On Non-separable Families of Positive Homothetic Convex Bodies

Károly Bezdek[1,2] · Zsolt Lángi[3]



Definition 1 Let $\mathbf{K}$ be a convex body in $\mathbb{R}^d$ and let $\mathcal{K} = \{\mathbf{x}_i + \tau_i \mathbf{K} \mid \mathbf{x}_i \in \mathbb{R}^d, \tau_i > 0, i = 1, 2, \ldots, n\}$, where $d \geq 2$ and $n \geq 2$. Assume that $\mathcal{K}$ is a non-separable family in short, an NS-family, meaning that every hyperplane intersecting $\mathrm{conv}(\bigcup \mathcal{K})$ intersects a member of $\mathcal{K}$ in $\mathbb{R}^d$, i.e., there is no hyperplane disjoint from $\bigcup \mathcal{K}$ that strictly separates some elements of $\mathcal{K}$ from all the other elements of $\mathcal{K}$ in $\mathbb{R}^d$. Then, let $\lambda(\mathcal{K}) > 0$ denote the smallest positive value λ such that a translate of $\lambda(\sum_{i=1}^{n} \tau_i)\mathbf{K}$ covers $\bigcup \mathcal{K}$.

Conjecture 1 (Goodman–Goodman [5]) *For every convex body* $\mathbf{K}$ *in* $\mathbb{R}^d$ *and every NS-family* $\mathcal{K} = \{\mathbf{x}_i + \tau_i \mathbf{K} \mid \mathbf{x}_i \in \mathbb{R}^d, \tau_i > 0, i = 1, 2, \ldots, n\}$ *the inequality* $\lambda(\mathcal{K}) \leq 1$ *holds for all* $d \geq 2$ *and* $n \geq 2$.

Counterexample to Conjecture 1 for card($\mathcal{K}$) ≥ 3 in $\mathbb{R}^d$, $d \geq 2$

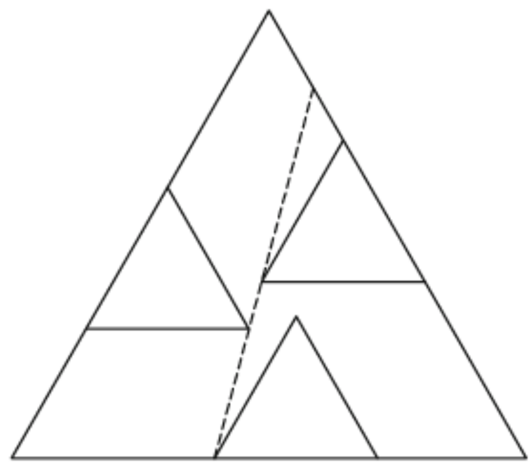


Fig. 1 A counterexample in the plane for three triangles

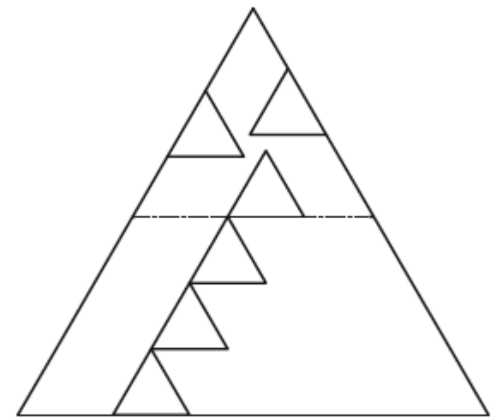


Fig. 2 A counterexample in the plane for n triangles

Example 1 Place three regular triangles $\mathcal{T} = \{\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3\}$ of unit side lengths into a regular triangle $\mathbf{T}$ of side length $2 + \frac{2}{\sqrt{3}} = 3.154700\ldots > 3$ such that

- each side of $\mathbf{T}$ contains a side of $\mathbf{T}_i$, for $i = 1, 2, 3$, respectively (cf. Fig. 1),
- for $i = 1, 2, 3$, the vertices of $\mathbf{T}_i$ contained in a side of $\mathbf{T}$ divide this side into three segments of lengths $\frac{2}{3} + \frac{1}{\sqrt{3}}$, 1, and $\frac{1}{3} + \frac{1}{\sqrt{3}}$, in counter-clockwise order.

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Remark 2 Let $d \geq 2$, and let λ_s^d denote the supremum of $\lambda(\mathcal{K})$, where $\mathcal{K}$ runs over the NS-families of finitely many positive homothetic d-simplices in $\mathbb{R}^d$. Then λ_s^d is a non-decreasing sequence of d.

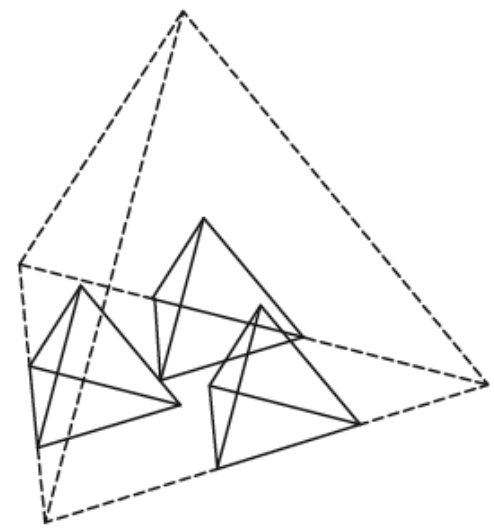

Fig. 3 A counterexample in $\mathbb{R}^3$ for three tetrahedra

Figure 3 shows how to extend the configuration in Example 1 to $\mathbb{R}^3$, implying that $\lambda_s^d \geq \lambda_s^2 \geq \frac{2}{3} + \frac{2}{3\sqrt{3}} = 1.0515\ldots > 1$ for all $d \geq 3$.

Remark 3 In fact, $\lambda_s^d = \sup_{\mathcal{K}} \lambda(\mathcal{K})$ for all $d \geq 2$, where $\mathcal{K}$ runs over the NS-families of finitely many positive homothetic copies of an arbitrary convex body $\mathbf{K}$ in $\mathbb{R}^d$.

Proof Clearly, $\lambda_s^d \leq \sup_{\mathcal{K}} \lambda(\mathcal{K})$. So, it is sufficient to show that for every convex body $\mathbf{K}$ in $\mathbb{R}^d$ and every NS-family $\mathcal{K} = \{\mathbf{x}_i + \tau_i \mathbf{K} \mid \mathbf{x}_i \in \mathbb{R}^d, \tau_i > 0, i = 1, 2, \ldots, n\}$ a translate of $\lambda_s^d (\sum_{i=1}^n \tau_i)\mathbf{K}$ covers $\bigcup \mathcal{K}$. Now, according to Lutwak's containment Theorem [6] if $\mathbf{K}_1$ and $\mathbf{K}_2$ are convex bodies in $\mathbb{R}^d$ such that every circumscribed simplex of $\mathbf{K}_2$ has a translate that covers $\mathbf{K}_1$, then $\mathbf{K}_2$ has a translate that covers $\mathbf{K}_1$. (Here a circumscribed simplex of $\mathbf{K}_2$ means a d-simplex of $\mathbb{R}^d$ that contains $\mathbf{K}_2$ such that each facet of the d-simplex meets $\mathbf{K}_2$.) Thus, if $\Delta(\mathbf{K})$ is a circumscribed simplex of $\mathbf{K}$, then $\lambda_s^d (\sum_{i=1}^n \tau_i)\Delta(\mathbf{K})$ is a circumscribed simplex of $\lambda_s^d (\sum_{i=1}^n \tau_i)\mathbf{K}$ and $\mathbf{x}_i + \tau_i \Delta(\mathbf{K})$ is a circumscribed simplex of $\mathbf{x}_i + \tau_i \mathbf{K}$ for all $i = 1, 2, \ldots, n$. Furthermore, $\{\mathbf{x}_i + \tau_i \Delta(\mathbf{K}) \mid \mathbf{x}_i \in \mathbb{R}^d, \tau_i > 0, i = 1, 2, \ldots, n\}$ is an NS-family and therefore $\lambda_s^d (\sum_{i=1}^n \tau_i)\Delta(\mathbf{K})$ has a translate that covers $\bigcup\{\mathbf{x}_i + \tau_i \Delta(\mathbf{K}) \mid \mathbf{x}_i \in \mathbb{R}^d, \tau_i > 0, i = 1, 2, \ldots, n\} \supseteq \bigcup \mathcal{K}$, which completes our proof via Lutwak's containment theorem. □

A Proof of Conjecture 1 for Centrally Symmetric Convex Bodies in $\mathbb{R}^d$, $d \geq 2$

Theorem 4 *For every* $\mathbf{o}$*-symmetric convex body* $\mathbf{K}_0$ *and every NS-family* $\mathcal{K} = \{\mathbf{x}_i + \tau_i \mathbf{K}_0 \mid \mathbf{x}_i \in \mathbb{R}^d, \tau_i > 0, i = 1, 2, \ldots, n\}$ *the inequality* $\lambda(\mathcal{K}) \leq 1$ *holds for all* $d \geq 2$ *and* $n \geq 2$.

Remark 5 If the positive homothetic convex bodies of $\mathcal{K} = \{\mathbf{x}_i + \tau_i \mathbf{K}_0 \mid \mathbf{x}_i \in \mathbb{R}^d, \tau_i > 0, i = 1, 2, \ldots, n\}$ have pairwise disjoint interiors with their centers $\{\mathbf{x}_i \mid i = 1, 2, \ldots, n\}$ lying on a line L in $\mathbb{R}^d$ such that the consecutive elements of $\mathcal{K}$ along L touch each other, then $\mathcal{K}$ is an NS-family with $\lambda(\mathcal{K}) = 1$.

Lemma 3 *Let* $\mathcal{F} = \{[x_i - \tau_i, x_i + \tau_i] \mid \tau_i > 0, i = 1, 2, \ldots, n\}$ *be a family of closed intervals in* $\mathbb{R}$ *such that* $\bigcup \mathcal{F}$ *is single closed interval in* $\mathbb{R}$. *Let* $x = \frac{\sum_{i=1}^n \tau_i x_i}{\sum_{i=1}^n \tau_i}$. *Then the interval* $[x - \sum_{i=1}^n \tau_i, x + \sum_{i=1}^n \tau_i]$ *covers* $\bigcup \mathcal{F}$.

Proof of Theorem 4 Let $\mathbf{x} = \frac{\sum_{i=1}^n \tau_i \mathbf{x}_i}{\sum_{i=1}^n \tau_i}$, and set $\mathbf{K}' = \mathbf{x} + (\sum_{i=1}^n \tau_i)\mathbf{K}_0$. We prove that $\mathbf{K}'$ covers $\bigcup \mathcal{K}$.

For any line L through the origin $\mathbf{o}$, let $\mathrm{proj}_L : \mathbb{R}^d \to L$ denote the orthogonal projection onto L, and let $h_{\mathcal{K}} : \mathbb{S}^{d-1} \to \mathbb{R}$ and $h_{\mathbf{K}'} : \mathbb{S}^{d-1} \to \mathbb{R}$ denote the support functions of $\mathrm{conv}(\bigcup \mathcal{K})$ and $\mathbf{K}'$, respectively. Then $\mathrm{proj}_L(\bigcup \mathcal{K})$ is a single interval, which, by Lemma 3, is covered by $\mathrm{proj}_L(\mathbf{K}')$. Thus, for any $\mathbf{u} \in \mathbb{S}^{d-1}$, we have that $h_{\mathcal{K}}(\mathbf{u}) \leq h_{\mathbf{K}'}(\mathbf{u})$, which readily implies that $\bigcup \mathcal{K} \subseteq \mathbf{K}'$. □

Upper Bounding $\lambda(\mathcal{K})$ in $\mathbb{R}^d$, $d \geq 2$

Theorem 3 *If $\mathcal{K} = \{\mathbf{x}_i + \tau_i \mathbf{K} \mid \mathbf{x}_i \in \mathbb{R}^d, \tau_i > 0, i = 1, 2, \ldots, n\}$ is an arbitrary NS-family of positive homothetic copies of the convex body $\mathbf{K}$ in $\mathbb{R}^d$, then*

$$\lambda(\mathcal{K}) \leq d$$

holds for all $n \geq 2$ and $d \geq 2$.

Problem 1 *Find $\sup_{\mathcal{K}} \lambda(\mathcal{K})$ for any given $d \geq 2$, where $\mathcal{K}$ runs over the NS-families of finitely many positive homothetic copies of an arbitrary convex body $\mathbf{K}$ in $\mathbb{R}^d$. In particular, is there an absolute constant $c > 0$ such that $\sup_{\mathcal{K}} \lambda(\mathcal{K}) \leq c$ holds for all $d \geq 2$?*

The results of this paper imply that $\frac{2}{3} + \frac{2}{3\sqrt{3}} = 1.0515\ldots \leq \sup_{\mathcal{K}} \lambda(\mathcal{K}) \leq d$ for all $d \geq 2$.

Problem 2 *Let $\mathbf{K}$ be a convex body in $\mathbb{R}^2$. If $\mathcal{F}$ is an NS-family of k positive homothetic copies of $\mathbf{K}$, with homothety ratios $\tau_1, \tau_2, \ldots, \tau_k$, respectively, and with $k \geq 4$, then prove or disprove that there is a translate of $(\frac{2}{3} + \frac{2}{3\sqrt{3}})(\sum_{i=1}^{k} \tau_i)\mathbf{K}$ containing $\mathcal{F}$.*

A Proof of Conjecture 1 for k-Impassable Families in $\mathbb{R}^d$ Whenever $0 \leq k \leq d-2$

Definition 2 Let $\mathcal{K} = \{\mathbf{x}_i + \tau_i \mathbf{K} \mid \mathbf{x}_i \in \mathbb{R}^d, \tau_i > 0, i = 1, 2, \ldots, n\}$ be a family of positive homothetic copies of the convex body $\mathbf{K}$ in $\mathbb{R}^d$ and let $0 \leq k \leq d-1$. We say that $\mathcal{K}$ is a *k-impassable arrangement*, in short, a *k-IP-family* if every k-dimensional affine subspace of $\mathbb{R}^d$ intersecting $\mathrm{conv}(\bigcup \mathcal{K})$ intersects an element of $\mathcal{K}$. Let $\lambda_k(\mathcal{K}) > 0$ denote the smallest positive value λ such that some translate of $\lambda(\sum_{i=1}^{n} \tau_i)\mathbf{K}$ covers $\bigcup \mathcal{K}$, where $\mathcal{K}$ is k-IP-family. A $(d-1)$-*IP-family* is simply called an *NS-family* and in that case $\lambda_{d-1}(\mathcal{K}) = \lambda(\mathcal{K})$.

Theorem 5 *Let* $\mathbf{K}$ *be a d-dimensional convex body and* $\mathcal{K} = \{\mathbf{K}_i = \mathbf{x}_i + \tau_i \mathbf{K} \mid \mathbf{x}_i \in \mathbb{R}^d, \tau_i > 0, i = 1, 2, \ldots, n\}$ *be a k-IP family of positive homothetic copies of* $\mathbf{K}$ *in* $\mathbb{R}^d$, *where* $0 \leq k \leq d-2$. *Then* $\mathrm{conv} \bigcup \mathcal{K}$ *slides freely in* $(\sum_{i=1}^{n} \tau_i)\mathbf{K}$ *(i.e.,* $\mathrm{conv} \bigcup \mathcal{K}$ *is a summand of* $(\sum_{i=1}^{n} \tau_i)\mathbf{K}$ *) and therefore* $\lambda_k(\mathcal{K}) \leq 1$.

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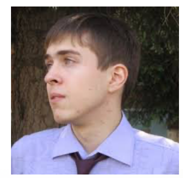
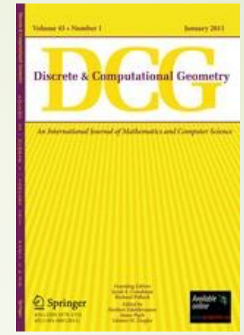
On the Circle Covering Theorem by A.W. Goodman and R.E. Goodman

Arseniy Akopyan[1] · Alexey Balitskiy[2,3,4] · Mikhail Grigorev[2]





Theorem 3.1 *Let $\mathcal{K}$ be a family of positive homothetic copies of a simplex $K \subset \mathbb{R}^d$ with homothety coefficients $\tau_1, \ldots, \tau_n > 0$. Suppose any hyperplane H (parallel to a facet of K) intersecting $\operatorname{conv} \bigcup \mathcal{K}$ intersects a member of $\mathcal{K}$. Then it is possible to cover $\bigcup \mathcal{K}$ by a translate of $\frac{d+1}{2}\left(\sum \tau_i\right)K$. Moreover, the factor $\frac{d+1}{2}$ cannot be improved.*

Let $K \subset \mathbb{R}^d$ be a (not necessarily centrally-symmetric) convex body containing the origin and let $K^\circ = \{p : \langle p, q\rangle \leq 1 \text{ for all } q \in K\}$ (where $\langle \cdot, \cdot \rangle$ stands for the standard inner product) be its polar body. We define the following *parameter of asymmetry*:

$$\sigma = \min_{q \in \text{int}\, K} \min\{\mu > 0 : (K - q) \subset -\mu(K - q)\}.$$

Lemma 2.2 (H. Minkowski and J. Radon) *Let K be a convex body in $\mathbb{R}^d$. Then $\sigma \leq d$, where σ denotes the parameter of asymmetry of K, defined above.*

Theorem 2.1 *Given a non-separable family of positive homothetic copies of (not necessarily centrally-symmetric) convex body $K \subset \mathbb{R}^d$ with homothety coefficients $\tau_1, \ldots, \tau_n > 0$, it is always possible to cover them by a translate of $\frac{\sigma+1}{2}\left(\sum \tau_i\right)K$.*

Part I/B – Non-separable arrangements in spherical spaces

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A CAP COVERING THEOREM

ALEXANDR POLYANSKII*

2.2. Non-separable arrangements in spherical spaces. In order to state the main results of this section we need to recall some definitions. A (closed) cap of spherical radius α, i.e., a (closed) spherical ball of radius α, for $0 \leq \alpha \leq \pi$, is the set of points with spherical distance at most α from a given point in $\mathbb{S}^{d-1} \subset \mathbb{E}^d$. A great sphere of $\mathbb{S}^{d-1}$ is an intersection of $\mathbb{S}^{d-1}$ with a hyperplane of $\mathbb{E}^d$ passing through the origin $\mathbf{o} \in \mathbb{E}^d$. Following the terminology of Polyanskii [Pol21], we say that a great sphere avoids a collection of caps in $\mathbb{S}^{d-1}$ if it does not intersect any cap of the collection. Finally, we say that a finite collection of caps is non-separable if it does not have a great sphere that avoids the caps such that on both sides of it there is at least one cap. Based on these concepts Polyanskii [Pol21] very recently proved the following extension of Theorem 2.1 to spherical spaces.

Theorem 2.8. *Let $\mathcal{F}$ be a non-separable family of caps of spherical radii $\alpha_1, \ldots, \alpha_n$ in $\mathbb{S}^{d-1}$, $d \geq 2$. If $\alpha_1 + \cdots + \alpha_n < \frac{\pi}{2}$, then $\mathcal{F}$ can be covered by a cap of radius $\alpha_1 + \cdots + \alpha_n$ in $\mathbb{S}^{d-1}$.*

It is shown in [Pol21] that Theorem 2.8 is equivalent to the following theorem. Polyanskii's proof of Theorem 2.9 uses ideas from [Bal21], [Bal91], and [Ban51]. Recall that a (closed) zone of width 2α with $0 \leq \alpha \leq \frac{\pi}{2}$ in $\mathbb{S}^{d-1}$ is the set of points with spherical distance at most α from a given great sphere in $\mathbb{S}^{d-1}$. Notice that a zone of width 2α with $0 < \alpha < \frac{\pi}{2}$ is not spherically convex in $\mathbb{S}^{d-1}$. However, the two connected components of the complement of any such zone, are spherically convex open caps in $\mathbb{S}^{d-1}$.

Theorem 2.9. *Let $\mathbf{Z}_1 \subset \mathbb{S}^{d-1}, \ldots, \mathbf{Z}_n \subset \mathbb{S}^{d-1}$, $d \geq 2$ be zones of width $2\alpha_1, \ldots, 2\alpha_n$, respectively, such that $\alpha_1 + \cdots + \alpha_n < \frac{\pi}{2}$. If $\mathbb{S}^{d-1} \setminus (\bigcup_{i=1}^n \mathbf{Z}_i)$ has at most one pair of antipodal open connected components, then $\bigcup_{i=1}^n \mathbf{Z}_i$ can be covered by a zone of width $2\alpha_1 + \cdots + 2\alpha_n$.*

According to [Pol21], Maxim Didid suggested to investigate the following more general problem, which he phrased as a conjecture. In what follows, a spherically convex body say, $\mathbf{K}$ of $\mathbb{S}^{d-1} \subset \mathbb{E}^d$ is the intersection of $\mathbb{S}^{d-1}$ with a d-dimensional closed convex cone of $\mathbb{E}^d$ different from $\mathbb{E}^d$. The inradius of $\mathbf{K}$ is the spherical radius of the largest cap contained in $\mathbf{K}$.

Conjecture 1. *Let* $\mathbf{Z}_1 \subset \mathbb{S}^{d-1}, \dots, \mathbf{Z}_n \subset \mathbb{S}^{d-1}$, $d > 2$ *be zones of width* $2\beta_1, \dots, 2\beta_n$, *respectively. If* $\mathbb{S}^{d-1} \setminus (\bigcup_{i=1}^n \mathbf{Z}_i)$ *consists of* $2m$ *spherically convex open connected components with inradii* $\gamma_1, \dots, \gamma_{2m}$, *respectively, then* $2\beta_1 + \dots 2\beta_n + \gamma_1 + \dots + \gamma_{2m} \geq \pi$.

Remark 1. *Bezdek and Schneider* [BS10] *proved that if a cap of spherical radius $\alpha \geq \frac{\pi}{2}$ is covered by a finite family of spherically convex bodies in $\mathbb{S}^{d-1}$, $d > 2$, then the sum of the inradii of the spherically convex bodies in the family is at least α. Clearly, this theorem implies Conjecture 1 for the case when $\beta_1 = \cdots = \beta_n = 0$, i.e., $\mathbf{Z}_1, \ldots, \mathbf{Z}_n$ are great spheres of $\mathbb{S}^{d-1}$.*

As Problem 7.3.5 of [Bez13] is still open, therefore we mention it here as well in connection with Remark 1.

Problem 2. *Prove or disprove that if a cap of spherical radius $0 < \alpha < \frac{\pi}{2}$ is covered by a finite family of spherically convex bodies in $\mathbb{S}^{d-1}$, $d > 2$, then the sum of the inradii of the spherically convex bodies in the family is at least α.*

Remark 2. *Jiang and Polyanskii* [JP17] *proved that if a finite family of zones covers $\mathbb{S}^{d-1}$, then the sum of the widths of the zones in the family is at least π. Ortega-Moreno* [OM21] *has found another proof, which was simplified by Zhao* [Zha22] *(see also* [GKP23]*). Clearly, this theorem implies Conjecture 1 for the case when $\gamma_1 = \cdots = \gamma_{2m} = 0$, i.e., $\mathbb{S}^{d-1} = \bigcup_{i=1}^{n} \mathbf{Z}_i$.*

Part II – Dense Totally Separable Packings

Part II/A – Results in 2D

- Finding the maximum density of TS-packings by congruent copies of a centrally symmetric convex domain

[FTFT73] G. Fejes Tóth and L. Fejes Tóth, *On totally separable domains*, Acta Math. Acad. Sci. Hungar. **24** (1973), 229–232. MR322690

3.1. Results in the Euclidean (resp., spherical) plane. The concept of totally separable packings was introduced by Fejes Tóth and Fejes Tóth [FTFT73] as follows.

Definition 2. *A packing $\mathcal{F}$ of convex domains in $\mathbb{E}^2$ is called a* totally separable packing, *in short, a* TS-packing, *if any two members of $\mathcal{F}$ can be separated by a line which is disjoint from the interiors of all members of $\mathcal{F}$.*

We shall use the following notation.

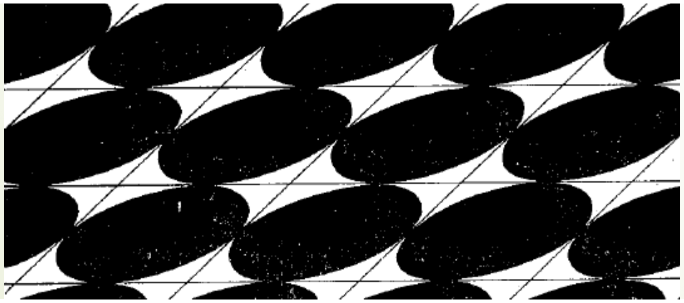
Definition 3. *Let* $\mathbf{K}$ *be a convex domain in* $\mathbb{E}^2$. *Then let* $\square^*(\mathbf{K})$ *(resp.,* $\square(\mathbf{K})$*) denote a minimal area circumscribed quadrilateral (resp., parallelogram) of* $\mathbf{K}$.

Fejes Tóth and Fejes Tóth [FTFT73] put forward the problem of finding the largest density of TS-packings by congruent copies of a given convex domain in $\mathbb{E}^2$. They have solved this problem for centrally symmetric convex domains as follows.

Theorem 3.1. *Let* $\mathbf{K}$ *be a convex domain in* $\mathbb{E}^2$ *and let* $\mathbf{Q} \subset \mathbb{E}^2$ *be a convex quadrilateral that contains* $n > 1$ *congruent copies of* $\mathbf{K}$ *forming a TS-packing in* $\mathbf{Q}$. *Then* $\mathrm{area}(\mathbf{Q}) \geq n \cdot \mathrm{area}(\square^*(\mathbf{K}))$.

According to a theorem of Dowker [Dow44] if $\mathbf{K} \subset \mathbb{E}^2$ is a centrally symmetric convex domain, then among the least area convex quadrilaterals containing $\mathbf{K}$ there is a parallelogram. Clearly, this observation and Theorem 3.1 imply

Corollary 3.2. *Let* $\mathbf{K}$ *be a centrally symmetric convex domain and let* $\mathcal{P}$ *be an arbitrary TS-packing by congruent copies of* $\mathbf{K}$. *Then for the density* $\delta(\mathcal{P})$ *of* $\mathcal{P}$ *(i.e., for the fraction* $\delta(\mathcal{P})$ *of* $\mathbb{E}^2$ *covered by the members of* $\mathcal{P}$*) we have that* $\delta(\mathcal{P}) \leq \frac{\mathrm{area}(\mathbf{K})}{\mathrm{area}(\square(\mathbf{K}))}$. *Here, equality is attained for the lattice TS-packing of* $\mathbf{K}$ *with fundamental parallelogram* $\square(\mathbf{K})$.



➨ Minimizing the convex hull of TS-packings by n>1 translates of a convex domain

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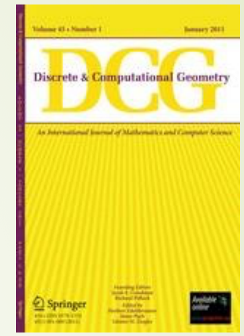
Bounds for Totally Separable Translative Packings in the Plane

Károly Bezdek[1,2] · Zsolt Lángi[3]



[BL20] K. Bezdek and Z. Lángi, *Bounds for totally separable translative packings in the plane*, Discrete Comput. Geom. **63** (2020), no. 1, 49–72. MR4045741

The line of research started in [FTFT73] has been continued by the author and Lángi in [BL20]. On the one hand, the following close relative of Corollary 3.2 was proved in [BL20].

Theorem 3.3. *If $\delta_{sep}(\mathbf{K})$ denotes the largest (upper) density of TS-packings by translates of the convex domain $\mathbf{K}$ in $\mathbb{E}^2$, then*

$$\delta_{sep}(\mathbf{K}) = \frac{\text{area}(\mathbf{K})}{\text{area}(\square(\mathbf{K}))}. \tag{3}$$

Remark 3. *It is worth pointing out that by (3) of Theorem 3.3, the densest TS-packing by translates of a convex domain is attained by a lattice packing.*

On the other hand, the following finite TS-packing analogue of Theorem 3.3 was proved by the author and Lángi in [BL20].

Theorem 3.4. *Let* $\mathcal{F} = \{\mathbf{c}_i + \mathbf{K} : i = 1, 2, \dots, n\}$ *be a TS-packing by* n *translates of the convex domain* $\mathbf{K}$ *in* $\mathbb{E}^2$. *Let* $\mathbf{C} = \operatorname{conv}\{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n\}$.

(3.4.1) Then we have

$$\operatorname{area}\left(\operatorname{conv}\left(\bigcup_{i=1}^{n}(\mathbf{c}_i + \mathbf{K})\right)\right) = \operatorname{area}(\mathbf{C} + \mathbf{K}) \geq \frac{2}{3}(n-1)\operatorname{area}(\Box(\mathbf{K})) + \operatorname{area}(\mathbf{K}) + \frac{1}{3}\operatorname{area}(\mathbf{C}).$$

(3.4.2) If $\mathbf{K}$ *or* $\mathbf{C}$ *is centrally symmetric, then*

$$\operatorname{area}(\mathbf{C} + \mathbf{K}) \geq (n-1)\operatorname{area}(\Box(\mathbf{K})) + \operatorname{area}(\mathbf{K}).$$

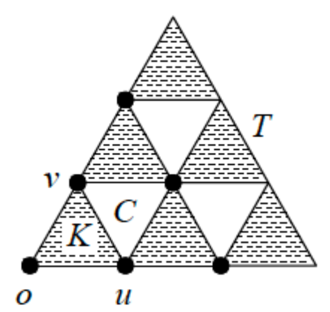


FIGURE 3. An example for equality in (3.4.1)

Remark 4. *We note that equality is attained in* (3.4.1) *of Theorem 3.4 for the following TS-packings by translates of a triangle (cf. Figure 3). Let* $\mathbf{K}$ *be a triangle, with the origin* $\mathbf{o}$ *at a vertex, and* $\mathbf{u}$ *and* $\mathbf{v}$ *being the position vectors of the other two vertices, and let* $\mathbf{T} = m\mathbf{K}$*, where* $m > 1$ *is an integer. Let* $\mathcal{F}$ *be the family consisting of the elements of the lattice packing* $\{i\mathbf{u} + j\mathbf{v} + \mathbf{K} : i, j, \in \mathbb{Z}\}$ *contained in* $\mathbf{T}$*. Then* $\mathcal{F}$ *is a TS-packing by* $n = \frac{m(m+1)}{2}$ *translates of* $\mathbf{K}$ *with* $\operatorname{conv}(\bigcup \mathcal{F}) = \mathbf{T} = \mathbf{C} + \mathbf{K}$*, where* $\mathbf{C} = (m-1)\mathbf{K}$*. Thus,* $\operatorname{area}(\mathbf{T}) = m^2\operatorname{area}(\mathbf{K}) = \left[\frac{2}{3}m(m+1) - \frac{1}{3} + \frac{1}{3}(m-1)^2\right]\operatorname{area}(\mathbf{K}) = \frac{4}{3}(n-1)\operatorname{area}(\mathbf{K}) + \operatorname{area}(\mathbf{K}) + \frac{1}{3}\operatorname{area}(\mathbf{C}) = \frac{2}{3}(n-1)\operatorname{area}(\Box(\mathbf{K})) + \operatorname{area}(\mathbf{K}) + \frac{1}{3}\operatorname{area}(\mathbf{C})$.

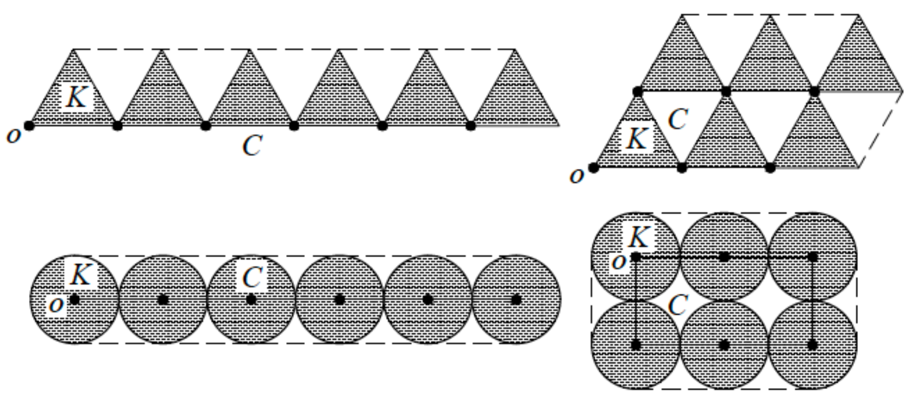


FIGURE 4. TS-packings by translates of a triangle and a unit disk for which equality is attained in (3.4.2) in Theorem 3.4

Remark 5. *In (3.4.2) of Theorem 3.4 equality can be attained in a variety of ways shown in Figure 4 for both cases namely, when* $\mathbf{C}$ *is centrally symmetric (and* $\mathbf{K}$ *is not centrally symmetric such as a triangle) and when* $\mathbf{K}$ *is centrally symmetric (such as a circular disk) without any assumption on the symmetry of* $\mathbf{C}$.

An analogue of Oler's inequality for translative TS-packings

The proofs of Theorems 3.3 and 3.4 in [BL20] are based on a translative TS-packing analogue of Oler's inequality [Ole61]. As it might be of independent interest, we state it as follows. First, we recall the necessary definitions.

If $\mathbf{K}$ is an $\mathbf{o}$-symmetric convex domain in $\mathbb{E}^2$, then let $|\cdot|_{\mathbf{K}}$ denote the *norm generated by* $\mathbf{K}$, i.e., let $|\mathbf{x}|_{\mathbf{K}} = \min\{\lambda : \mathbf{x} \in \lambda\mathbf{K}\}$ for any $\mathbf{x} \in \mathbb{E}^2$. The distance between the points $\mathbf{p}$ and $\mathbf{q}$ of $\mathbb{E}^2$ measured in the norm $|\cdot|_{\mathbf{K}}$ is denoted by $|\mathbf{p}-\mathbf{q}|_K$. For the sake of simplicity, the Euclidean distance between the points $\mathbf{p}$ and $\mathbf{q}$ of $\mathbb{E}^2$ is denoted by $|\mathbf{p}-\mathbf{q}|$.

If $P = \bigcup_{i=1}^{n}[\mathbf{x}_{i-1}, \mathbf{x}_i]$ is a polygonal curve in $\mathbb{E}^2$ with $[\mathbf{x}_{i-1}, \mathbf{x}_i]$ standing for the closed line segment connecting $\mathbf{x}_{i-1}$ and $\mathbf{x}_i$, and $\mathbf{K}$ is an $\mathbf{o}$-symmetric convex domain in $\mathbb{E}^2$, then the *Minkowski length of P* is defined as $M_{\mathbf{K}}(P) = \sum_{i=1}^{n} |\mathbf{x}_i - \mathbf{x}_{i-1}|_{\mathbf{K}}$. Based on this and using approximation by polygonal curves one can define the Minkowski length $M_{\mathbf{K}}(G)$ of any rectifiable curve $G \subseteq \mathbb{E}^2$ in the norm $|\cdot|_{\mathbf{K}}$.

Definition 4. *A closed polygonal curve $P = \bigcup_{i=1}^{m}[\mathbf{x}_{i-1}, \mathbf{x}_i]$, where $\mathbf{x}_0 = \mathbf{x}_m$, is called* permissible *if there is a sequence of simple closed polygonal curves $P^n = \bigcup_{i=1}^{m}[\mathbf{x}_{i-1}^n, \mathbf{x}_i^n]$, where $\mathbf{x}_0^n = \mathbf{x}_m^n$, satisfying $\mathbf{x}_i^n \to \mathbf{x}_i$ for every value of i. The interior* $\mathrm{int}P$ *is defined as* $\lim_{n\to\infty} \mathrm{int}P^n$.

One of the main results of [BL20] is the following translative TS-packing analogue of Oler's inequality [Ole61].

Theorem 3.6. *Let* $\mathbf{K}$ *be an* $\mathbf{o}$*-symmetric convex domain in* $\mathbb{E}^2$*. Let*

$$\mathcal{F} = \{\mathbf{x}_i + \mathbf{K} : i = 1, 2, \ldots, n\}$$

be a TS-packing by n *translates of* $\mathbf{K}$ *in* $\mathbb{E}^2$*, and set* $X = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$*. Furthermore, let* Π *be a permissible closed polygonal curve with the following properties:*

(1) the vertices of Π *are points of* X

and

(2) $X \subseteq \Pi^*$ *with* $\Pi^* = \Pi \cup \mathrm{int}\Pi$.

Then

$$\frac{\mathrm{area}(\Pi^*)}{\mathrm{area}\left(\Box(\mathbf{K})\right)} + \frac{M_{\mathbf{K}}(\Pi)}{4} + 1 \geq n. \tag{4}$$

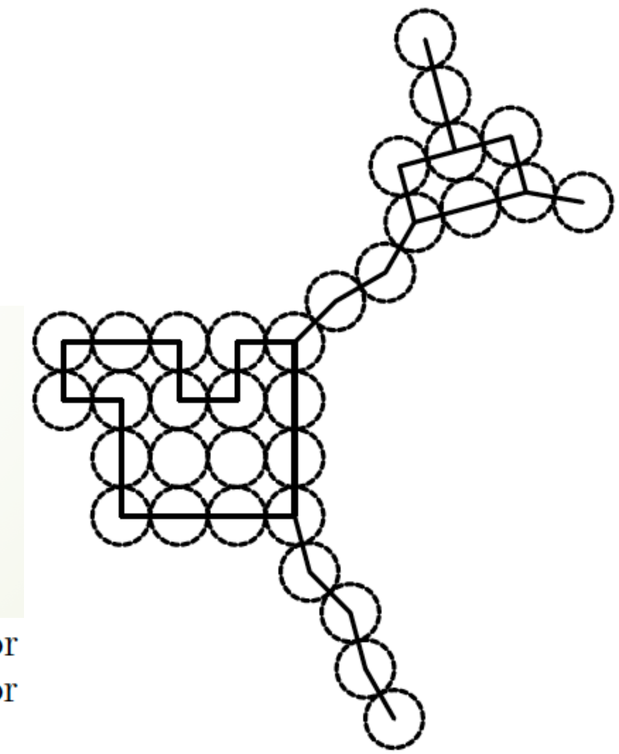


FIGURE 5. A TS-packing of translates of K (with K being a circular disk for the sake of simplicity), which satisfies the conditions in Theorem 3.6 and for which there is equality in (4) of Theorem 3.6.

As an application of Theorem 3.6, we outline the proof of (3.4.2) of Theorem 3.4 for centrally symmetric $\mathbf{K}$ following [BL20]. It goes as follows. Note that bd$\mathbf{C}$ satisfies the conditions in Theorem 3.6, and thus, we have

$$\frac{\mathrm{area}(\mathbf{C})}{\mathrm{area}\left(\Box(\mathbf{K})\right)} + \frac{M_{\mathbf{K}}(\mathrm{bd}\mathbf{C})}{4} + 1 \geq n.$$

Next, recall the following inequality from [BL20].

Lemma 3.7. *Let* $\mathbf{K}$ *be a convex domain in* $\mathbb{E}^2$ *and let* $\mathbf{Q}$ *be a convex polygon. Furthermore, let* $A(\mathbf{Q},\mathbf{K})$ *denote the mixed area of* $\mathbf{Q}$ *and* $\mathbf{K}$. *If* $\mathbf{K}$ *is centrally symmetric, then*

$$\frac{8A(\mathbf{Q},\mathbf{K})}{\mathrm{area}\left(\Box(\mathbf{K})\right)} \geq M_{\mathbf{K}}(\mathrm{bd}\mathbf{Q}).$$

Thus, Lemma 3.7 yields that

$$\frac{\mathrm{area}(\mathbf{C})}{\mathrm{area}\left(\Box(\mathbf{K})\right)} + \frac{2A(\mathbf{C},\mathbf{K})}{\mathrm{area}\left(\Box(\mathbf{K})\right)} + 1 \geq n.$$

From this, it follows that

$$\mathrm{area}\left(\mathrm{conv}\left(\bigcup_{i=1}^{n}(\mathbf{c}_i+\mathbf{K})\right)\right) = \mathrm{area}(\mathbf{C}+\mathbf{K}) = \mathrm{area}(\mathbf{C}) + 2A(\mathbf{C},\mathbf{K}) + \mathrm{area}(\mathbf{K}) \geq$$
$$(n-1)\mathrm{area}\left(\Box(\mathbf{K})\right) + \mathrm{area}(\mathbf{K}),$$

finishing the proof of (3.4.2) of Theorem 3.4 for centrally symmetric $\mathbf{K}$.

The separable Tammes problem

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From the Separable Tammes Problem to Extremal Distributions of Great Circles in the Unit Sphere

Károly Bezdek[1,2] · Zsolt Lángi[3]

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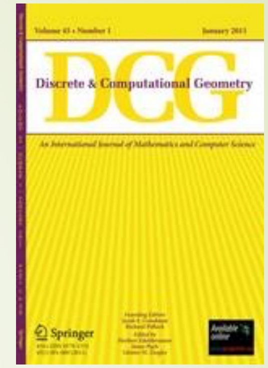


Definition 5. *A family of spherical caps of $\mathbb{S}^2$ is called a totally separable packing in short, a TS-packing if any two spherical caps can be separated by a great circle of $\mathbb{S}^2$ which is disjoint from the interior of each spherical cap in the packing.*

The analogue of the Tammes problem for TS-packings, called the *separable Tammes problem*, was raised in [BL22].

Problem 4. *For given $k > 1$ find the largest $r > 0$ such that there exists a TS-packing of k spherical caps with angular radius r in $\mathbb{S}^2$. Let us denote this r by $r_{\mathrm{STam}}(k, \mathbb{S}^2)$.*

Remark 8. *It was noted in* [BL22] *that $r_{\mathrm{STam}}(2k' - 1, \mathbb{S}^2) = r_{\mathrm{STam}}(2k', \mathbb{S}^2)$ holds for any integer $k' > 1$.*

The following TS-packings are special solutions of the separable Tammes problem ([BL22]).

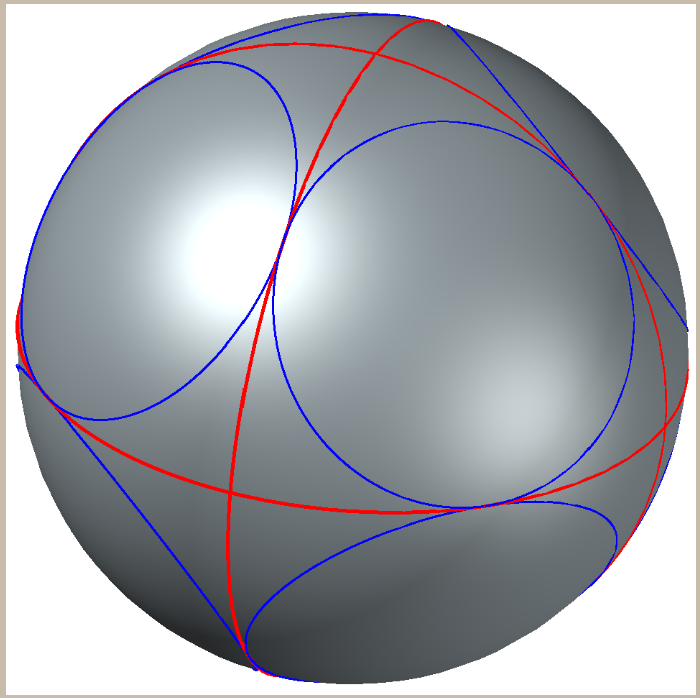


Figure 1: An octahedral TS-packing in $\mathbb{S}^2$.

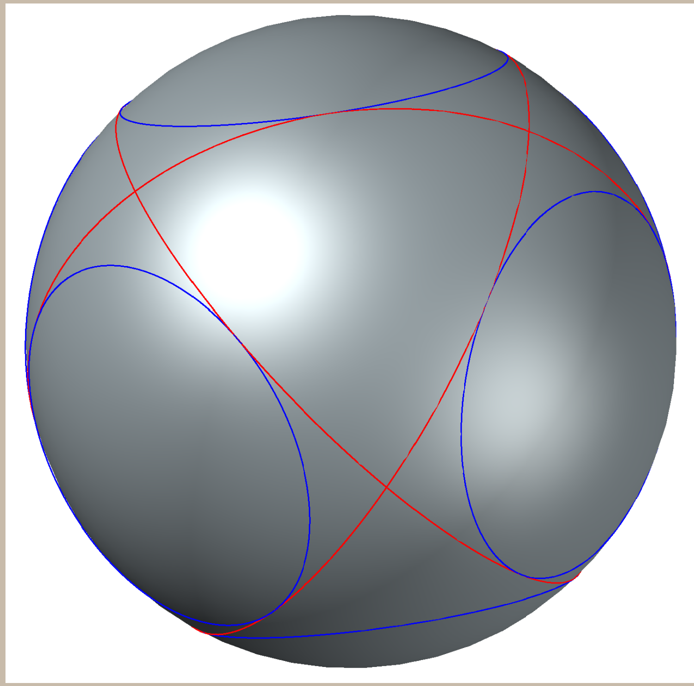


Figure 2: A cuboctahedral TS-packing in $\mathbb{S}^2$.

eight spherical caps of radius $\arcsin \frac{1}{\sqrt{3}} (\approx 35.26^\circ)$

six spherical caps of radius $\arctan \frac{3}{4} \approx 36.87^\circ$

On the one hand, it is easy to check that $r_{\mathrm{STam}}(2, \mathbb{S}^2) = \frac{\pi}{2}(= 90^\circ)$ and $r_{\mathrm{STam}}(3, \mathbb{S}^2) = r_{\mathrm{STam}}(4, \mathbb{S}^2) = \frac{\pi}{4}(= 45^\circ)$. On the other hand, Problem 4 is solved for $k = 5, 6, 7, 8$ in [BL22].

Definition 6. *Let $k > 1$ be fixed. A TS-packing of k spherical caps of radius $r_{\mathrm{STam}}(k, \mathbb{S}^2)$ is called* k-optimal.

Theorem 3.8. *For $5 \leq k \leq 6$ we have $r_{\mathrm{STam}}(k, \mathbb{S}^2) = \arctan \frac{3}{4}$, and any k-optimal TS-packing is a subfamily of a cuboctahedral TS-packing. Furthermore, for $7 \leq k \leq 8$ we have $r_{\mathrm{STam}}(k, \mathbb{S}^2) = \arcsin \frac{1}{\sqrt{3}}$, and any k-optimal TS-packing is a subfamily of an octahedral TS-packing.*

The value of $r_{\mathrm{STam}}(k, \mathbb{S}^2)$ is not known for any $k > 8$, which leads to

Problem 5. *Compute the exact value of $r_{\mathrm{STam}}(k, \mathbb{S}^2)$ for some small integers $k > 8$.*

On the other hand, the author and Lángi [BL22] lower and upper bounded $r_{\mathrm{STam}}(k, \mathbb{S}^2)$ for any (sufficiently large value of) k as follows.

Theorem 3.9.

(i) $r_{\mathrm{STam}}(k, \mathbb{S}^2) \leq \arccos \frac{1}{\sqrt{2}\sin\left(\frac{k}{k-2}\frac{\pi}{4}\right)}$ *for all* $k \geq 5$. *In particular,* $r_{\mathrm{STam}}(8, \mathbb{S}^2) = \arccos\sqrt{\frac{2}{3}} = \arcsin\frac{1}{\sqrt{3}} (\approx 35.26°)$.

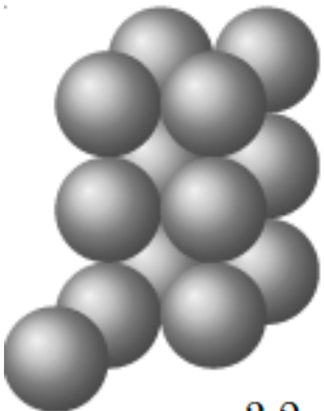
(ii) *For any sufficiently large value of* k, *we have* $r_{\mathrm{STam}}(k, \mathbb{S}^2) \geq \frac{0.793}{\sqrt{k}}$.

Notice that the upper bound on $r_{\mathrm{STam}}(k, \mathbb{S}^2)$ stated in Part (i) is about $\sqrt{\frac{\pi}{k}} \approx \frac{1.772}{\sqrt{k}}$ as $k \to +\infty$, which is of the same order of magnitude as the order of magnitude of the lower bound given in Part (ii).

Problem 6. *Does the limit* $\lim_{k\to+\infty} \sqrt{k} \cdot r_{\mathrm{STam}}(k, \mathbb{S}^2)$ *exist?*

Part II/B – Results in dimensions >2

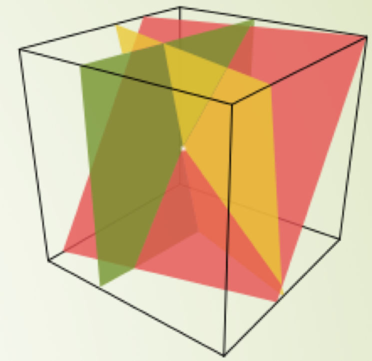
- Dense TS-packings by unit balls in 3D



Acta Math. Hung.
51 (3—4) (1988), 363—364.

ON TOTALLY SEPARABLE PACKINGS OF EQUAL BALLS

G. KERTÉSZ (Budapest)



3.2. **Results in dimensions > 2.** It is natural to extend the concept of TS-packings discussed in Definition 2 to higher dimensions as follows.

Definition 7. *A packing $\mathcal{F}$ of convex bodies in $\mathbb{E}^d$, $d > 2$ is called a* totally separable packing*, in short, a* TS-packing*, if any two members of $\mathcal{F}$ can be separated by a hyperplane of $\mathbb{E}^d$ which is disjoint from the interiors of all members of $\mathcal{F}$.*

An elegant paper of Kertész [Ker88] shows that the (upper) density of any TS-packing of unit diameter balls in $\mathbb{E}^3$ is at most $\frac{\pi}{6}$ with equality for the lattice packing of unit diameter balls having integer coordinates. Actually, Kertész [Ker88] proved the following stronger result: If a cube of volume $V > 0$ in $\mathbb{E}^3$ contains a TS-packing of $N > 1$ balls or radius $r > 0$, then $V \geq 8Nr^3$. In fact, it is not hard to see that Kertész's method of proof from [Ker88] implies the following even stronger result.

Theorem 3.10. *If a cube of volume $V > 0$ in $\mathbb{E}^3$ is partitioned into $N > 1$ convex cells by N-1 successive plane cuts (just one cell being divided by each cut) such that each convex cell contains a ball of radius $r > 0$, then the sum of the surface areas of the N convex cells is at least $24Nr^2$ and therefore $V \geq \frac{r}{3}(24Nr^2) = 8Nr^3$.*

It would be very interesting to find analogues of Theorem 3.10 in higher dimensions.

Problem 7. *Prove or disprove that if a d-dimensional cube of volume $V > 0$ in $\mathbb{E}^d$, $d > 3$ is partitioned into $N > 1$ convex cells by $N - 1$ successive hyperplane cuts (just one cell being divided by each cut) such that each convex cell contains a ball of radius $r > 0$, then $V \geq 2^d N r^d$.*

Remark 9. *A positive answer to Problem 7 would imply that the (upper) density of any TS-packing of unit diameter balls in $\mathbb{E}^d$, $d > 3$ is at most $\frac{\kappa_d}{2^d}$ with equality for the lattice packing of unit diameter balls having integer coordinates in $\mathbb{E}^d$.*

➨ Minimal ρ-separable packings

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https://doi.org/10.1007/s00605-018-1166-y

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Minimizing the mean projections of finite ρ-separable packings

Károly Bezdek[1,2] · Zsolt Lángi[3]

Next, recall the following elegant theorem of Böröczky Jr. [Bör94]: Consider the convex hull Q of n non-overlapping translates of an arbitrary convex body C in $\mathbb{E}^d$ with n being sufficiently large. If Q has minimal mean i-dimensional projection for given i with $1 \leq i < d$, then Q is approximately a d-dimensional ball. The author and Lángi [BL19] proved an extension of this theorem to the so-called ρ-separable translative packings of convex bodies in $\mathbb{E}^d$. In short, one can regard ρ-separable packings (for $\rho \geq 3$) as packings that are locally totally separable. In what follows, we define the concept of ρ-separable translative packings using [BL19] and then state the main result of [BL19].

Definition 8. *Let* $\mathbf{C}$ *be an* $\mathbf{o}$*-symmetric convex body of* $\mathbb{E}^d$*. Furthermore, let* $\|\cdot\|_{\mathbf{C}}$ *denote the norm generated by* $\mathbf{C}$*, i.e., let* $\|\mathbf{x}\|_{\mathbf{C}} := \inf\{\lambda \mid \mathbf{x} \in \lambda\mathbf{C}\}$ *for any* $\mathbf{x} \in \mathbb{E}^d$*. Now, let* $\rho \geq 1$*. We say that the packing*

$$\mathcal{P}_{\text{sep}} := \{\mathbf{c}_i + \mathbf{C} \mid i \in I \text{ with } \|\mathbf{c}_j - \mathbf{c}_k\|_{\mathbf{C}} \geq 2 \text{ for all } j \neq k \in I\}$$

of (finitely or infinitely many) non-overlapping translates of $\mathbf{C}$ *with centers* $\{\mathbf{c}_i \mid i \in I\}$ *is a* ρ*-separable packing in* $\mathbb{E}^d$ *if for each* $i \in I$ *the finite packing* $\{\mathbf{c}_j + \mathbf{C} \mid \mathbf{c}_j + \mathbf{C} \subseteq \mathbf{c}_i + \rho\mathbf{C}\}$ *is a TS-packing (in* $\mathbf{c}_i + \rho\mathbf{C}$*). Finally, let* $\delta_{\text{sep}}(\rho, \mathbf{C})$ *denote the largest upper density of all* ρ*-separable translative packings of* $\mathbf{C}$ *in* $\mathbb{E}^d$*, i.e., let*

$$\delta_{\text{sep}}(\rho, \mathbf{C}) := \sup_{\mathcal{P}_{\text{sep}}} \left(\limsup_{\lambda \to +\infty} \frac{\sum_{\mathbf{c}_i + \mathbf{C} \subset \mathbf{W}^d_\lambda} \mathrm{vol}_d(\mathbf{c}_i + \mathbf{C})}{\mathrm{vol}_d(\mathbf{W}^d_\lambda)} \right),$$

where $\mathbf{W}^d_\lambda$ *denotes the d-dimensional cube of edge length* 2λ *centered at* $\mathbf{o}$ *in* $\mathbb{E}^d$ *having edges parallel to the coordinate axes of* $\mathbb{E}^d$*.*

Recall that the mean i-dimensional projection $M_i(\mathbf{C})$ $(i = 1, 2, \ldots, d-1)$ of the convex body $\mathbf{C}$ in $\mathbb{E}^d$, can be expressed ([Sch14]) with the help of a mixed volume via the formula

$$M_i(\mathbf{C}) = \frac{\kappa_i}{\kappa_d} V(\overbrace{\mathbf{C}, \ldots, \mathbf{C}}^{i}, \overbrace{\mathbf{B}^d, \ldots, \mathbf{B}^d}^{d-i}),$$

where κ_d is the volume of $\mathbf{B}^d$ in $\mathbb{E}^d$. Note that $M_i(\mathbf{B}^d) = \kappa_i$, and the surface volume of $\mathbf{C}$ is $\mathrm{svol}_{d-1}(\mathbf{C}) = \frac{d\kappa_d}{\kappa_{d-1}} M_{d-1}(\mathbf{C})$ and in particular, $\mathrm{svol}_{d-1}(\mathbf{B}^d) = d\kappa_d$. Set $M_d(\mathbf{C}) := \mathrm{vol}_d(\mathbf{C})$. Finally, let $R(\mathbf{C})$ (resp., $r(\mathbf{C})$) denote the circumradius (resp., inradius) of the convex body $\mathbf{C}$ in $\mathbb{E}^d$, which is the radius of the smallest (resp., a largest) ball that contains (resp., is contained in) $\mathbf{C}$. The following is the main result of [BL19].

Theorem 3.11. *Let $d \geq 2$, $1 \leq i \leq d-1$, $\rho \geq 1$, and let $\mathbf{Q}$ be the convex hull of a ρ-separable packing of n translates of the $\mathbf{o}$-symmetric convex body $\mathbf{C}$ in $\mathbb{E}^d$ such that $M_i(\mathbf{Q})$ is minimal and $n \geq \frac{4^d d^{4d}}{\delta_{\mathrm{sep}}(\rho,\mathbf{C})^{d-1}} \cdot \left(\rho \frac{R(\mathbf{C})}{r(\mathbf{C})}\right)^d$. Then*

$$\frac{r(\mathbf{Q})}{R(\mathbf{Q})} \geq 1 - \frac{\omega}{n^{\frac{2}{d(d+3)}}}, \tag{5}$$

for $\omega = \lambda(d)\left(\frac{\rho R(\mathbf{C})}{r(\mathbf{C})}\right)^{\frac{2}{d+3}}$, where $\lambda(d)$ depends only on the dimension d. In addition,

$$M_i(\mathbf{Q}) = \left(1 + \frac{\sigma}{n^{\frac{1}{d}}}\right) M_i(\mathbf{B}^d) \left(\frac{\mathrm{vol}_d(\mathbf{C})}{\delta_{\mathrm{sep}}(\rho, \mathbf{C})\kappa_d}\right)^{\frac{i}{d}} \cdot n^{\frac{i}{d}},$$

where $-\frac{2.25 R(\mathbf{C})\rho d i}{r(\mathbf{C})\delta_{\mathrm{sep}}(\rho,\mathbf{C})} \leq \sigma \leq \frac{2.1 R(\mathbf{C})\rho i}{r(\mathbf{C})\delta_{\mathrm{sep}}(\rho,\mathbf{C})}$.

Remark 11. *It is worth restating Theorem 3.11 as follows: Consider the convex hull $\mathbf{Q}$ of n non-overlapping translates of an arbitrary $\mathbf{o}$-symmetric convex body $\mathbf{C}$ forming a ρ-separable packing in $\mathbb{E}^d$ with n being sufficiently large. If $\mathbf{Q}$ has minimal mean i-dimensional projection for given i with $1 \leq i < d$, then $\mathbf{Q}$ is approximately a d-dimensional ball.*

Problem 8. *Let $d \geq 2$, $1 \leq i \leq d-1$, and let $\mathbf{C}$ be an $\mathbf{o}$-symmetric convex body in $\mathbb{E}^d$. Does the analogue of Theorem 3.11 hold for translative TS-packings of $\mathbf{C}$ in $\mathbb{E}^d$?*

Remark 12. *The nature of the question analogue to Theorem 3.11 on minimizing $M_d(\mathbf{Q}) = \mathrm{vol}_d(\mathbf{Q})$ is very different. Namely, recall that Betke and Henk* [BH98] *proved L. Fejes Tóth's sausage conjecture for $d \geq 42$ according to which the smallest volume of the convex hull of n non-overlapping unit balls in $\mathbb{E}^d$ is obtained when the n unit balls form a sausage, that is, a linear packing. As linear packings of unit balls are ρ-separable, therefore the above theorem of Betke and Henk applies to ρ-separable packings of unit balls in $\mathbb{E}^d$ for all $\rho \geq 1$ and $d \geq 42$.*

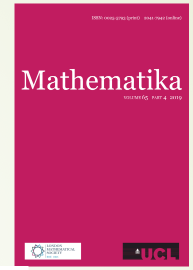
We close this section with the following conjecture, which has already been proved for $d = 2$ (see (3.4.2) of Theorem 3.4) as well as for all $d \geq 42$ (see Remark 12).

Conjecture 2. *The volume of the convex hull of an arbitrary TS-packing of $N > 1$ unit balls in $\mathbb{E}^d$ with $3 \leq d \leq 41$ is at least as large as the volume of the convex hull of N non-overlapping unit balls with their centers lying on a line segment of length 2(N-1).*

ON CONTACT NUMBERS OF LOCALLY SEPARABLE UNIT SPHERE PACKINGS

Mathematika 67 (2021) 714–729 doi:10.1112/mtk.12102

KÁROLY BEZDEK



Definition 9. *The contact graph $G(\mathcal{P})$ of a packing $\mathcal{P}$ of convex bodies in $\mathbb{E}^d$, $d > 1$ is the simple graph whose vertices correspond to the members of the packing, and whose two vertices are connected by an edge if the two members touch each other. The number of edges of $G(\mathcal{P})$ is called the contact number $c(\mathcal{P})$ of $\mathcal{P}$.*

The concept of locally separable (sphere) packing was introduced in [Bez21].

Definition 10. *We call the packing $\mathcal{P}$ of convex bodies in $\mathbb{E}^d$, $d > 1$ a locally separable packing, in short, an LS-packing if each member of $\mathcal{P}$ together with the members of $\mathcal{P}$ that are tangent to it form a TS-packing.*

Clearly, any TS-packing is also an LS-packing, but not necessarily the other way around. Moreover, it is worth noting that any ρ-separable packing by translates of a convex body for $\rho = 3$ is a translative LS-packing and vice versa.

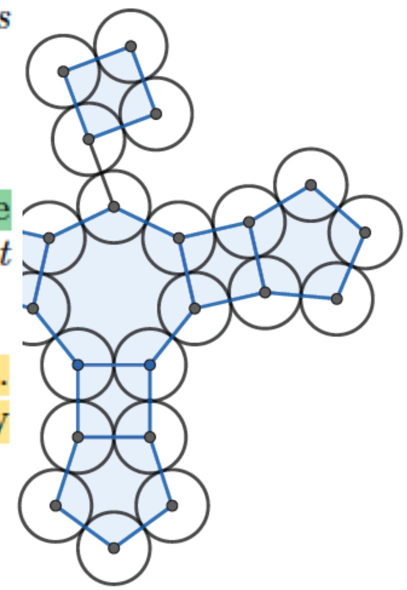


Figure 1 (colour online): An LS-packing of unit disks which is not a TS-packing.

Bounding the contact numbers of LS-packings of unit diameter disks

Theorem 4.1. *Let $\mathcal{P}^*$ be an arbitrary LS-packing of $n > 1$ unit diameter disks in $\mathbb{E}^2$. Then*

$$c(\mathcal{P}^*) \leq \lfloor 2n - 2\sqrt{n} \rfloor.$$

Futhermore, suppose that $\mathcal{P}$ is an LS-packing of n unit diameter disks with $c(\mathcal{P}) = \lfloor 2n - 2\sqrt{n} \rfloor$, $n \geq 4$ in $\mathbb{E}^2$. Let $G_c(\mathcal{P})$ denote the contact graph of $\mathcal{P}$ embedded in $\mathbb{E}^2$ such that the vertices are the center points of the unit diameter disks of $\mathcal{P}$ and the edges are line segments of unit length each connecting two center points if and only if the unit diameter disks centered at those two points touch each other. Then either $G_c(\mathcal{P})$ is the contact graph of the LS-packing of 7 unit diameter disks shown in Figure 8 or

(i) $G_c(\mathcal{P})$ is 2-connected whose internal faces (i.e., faces different from its external face) form an edge-to-edge connected family of unit squares called a polyomino of an isometric copy of the integer lattice $\mathbb{Z}^2$ in $\mathbb{E}^2$ (see the first packing in Figure 8) or

(ii) $G_c(\mathcal{P})$ is 2-connected whose internal faces are unit squares forming a polyomino of an isometric copy of the integer lattice $\mathbb{Z}^2$ in $\mathbb{E}^2$ with the exception of one internal face which is a pentagon adjacent along (at least) three consecutive sides to the external face of $G_c(\mathcal{P})$ and along (at most) two consecutive sides to the polyomino (see the second packing in Figure 9) or

(iii) $G_c(\mathcal{P})$ possesses a degree one vertex on the boundary of its external face such that deleting that vertex together with the edge adjacent to it yields a 2-connected graph whose internal faces are unit squares forming a polyomino of an isometric copy of the integer lattice $\mathbb{Z}^2$ in $\mathbb{E}^2$ (see the first packing in Figure 9).

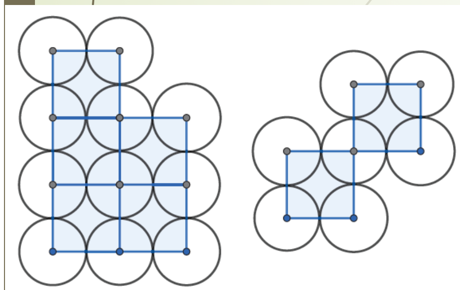


FIGURE 8.

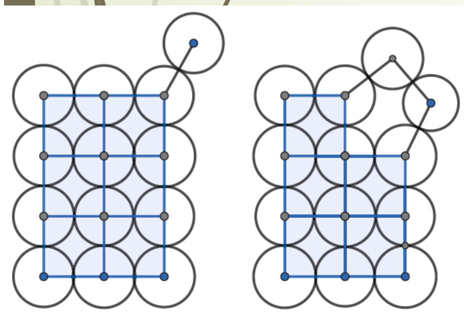


FIGURE 9.

Bounding the contact numbers of LS-packings of unit balls in high dimensions

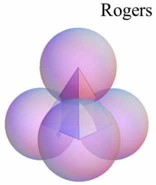
In higher dimensions we know much less. At present the best upper bound for contact numbers of LS-packings of congruent balls in $\mathbb{E}^d$, $d \geq 3$ is the one published in [Bez21]. We shall need the following notation. Let $\mathcal{P} := \{\mathbf{B}^d[\mathbf{c}_i, 1] | i \in I\}$ be an arbitrary (finite or infinite) packing of unit balls in $\mathbb{E}^d$, $d \geq 3$ and let $\mathbf{V}_i := \{\mathbf{x} \in \mathbb{E}^d | \ \|\mathbf{x} - \mathbf{c}_i\| \leq \|\mathbf{x} - \mathbf{c}_j\| \text{ for all } j \neq i, j \in I\}$ denote the Voronoi cell assigned to $\mathbf{B}^d[\mathbf{c}_i, 1]$ for $i \in I$. Recall ([Rog64]) that the Voronoi cells $\{\mathbf{V}_i | i \in I\}$ form a face-to-face tiling of $\mathbb{E}^d$. Then let the largest density of the unit ball $\mathbf{B}^d[\mathbf{c}_i, 1]$ in its truncated Voronoi cell $\mathbf{V}_i \cap \mathbf{B}^d[\mathbf{c}_i, \sqrt{d}]$ be denoted by $\hat{\delta}_d$, i.e., let $\hat{\delta}_d := \sup_{\mathcal{P}} \left(\sup_{i \in I} \frac{\omega_d}{\mathrm{vol}_d\left(\mathbf{V}_i \cap \mathbf{B}^d[\mathbf{c}_i, \sqrt{d}]\right)} \right)$, where $\mathcal{P}$ runs through all possible unit ball packings of $\mathbb{E}^d$. We are now ready to state the upper bound from [Bez21].

Theorem 4.3. *Let $\mathcal{P}$ be an arbitrary LS-packing of $n > 1$ unit balls in $\mathbb{E}^d$, $d \geq 3$. Then*

$$c(\mathcal{P}) \leq \left\lfloor dn - \left(d^{-\frac{d-3}{2}} \hat{\delta}_d^{-\frac{d-1}{d}} \right) n^{\frac{d-1}{d}} \right\rfloor . \tag{6}$$

Remarks:

Rogers bound



Remark 13. *Recall the following classical result of Rogers* [Rog58] *(which was rediscovered by Baranovskii* [Bar64] *and extended to spherical and hyperbolic spaces by Böröczky* [Bör78]*): Let* $\mathcal{P} := \{\mathbf{B}^d[\mathbf{c}_i, 1] | i \in I\}$ *be an arbitrary packing of unit balls in* $\mathbb{E}^d$, $d > 1$ *with* $\mathbf{V}_i$ *standing for the Voronoi cell assigned to* $\mathbf{B}^d[\mathbf{c}_i, 1]$ *for* $i \in I$*. Furthermore, take a regular d-dimensional simplex of edge length* 2 *in* $\mathbb{E}^d$ *and then draw a d-dimensional unit ball around each vertex of the simplex. Finally, let* σ_d *denote the ratio of the volume of the portion of the simplex covered by balls to the volume of the simplex. Then* $\frac{\omega_d}{\mathrm{vol}_d\left(\mathbf{V}_i \cap \mathbf{B}^d[\mathbf{c}_i, \sqrt{\frac{2d}{d+1}}]\right)} \leq \sigma_d$ *holds for all* $i \in I$ *and therefore* $\hat{\delta}_d \leq \sigma_d$ *for all* $d \geq 3$*. The latter inequality and (6) yield that if* $\mathcal{P}$ *is an arbitrary LS-packing of* $n > 1$ *unit balls in* $\mathbb{E}^d$, $d \geq 3$*, then* $c(\mathcal{P}) \leq \left\lfloor dn - \left(d^{-\frac{d-3}{2}} \hat{\delta}_d^{-\frac{d-1}{d}}\right) n^{\frac{d-1}{d}} \right\rfloor \leq \left\lfloor dn - \left(d^{-\frac{d-3}{2}} \sigma_d^{-\frac{d-1}{d}}\right) n^{\frac{d-1}{d}} \right\rfloor$*, where* $\sigma_d \sim \frac{d}{e} 2^{-\frac{1}{2}d}$ *(*[Rog58]*).*

Remark 15. *The density upper bound* σ_3 *of Rogers has been improved by Hales* [Hal12] *as follows: If* $\mathcal{P} := \{\mathbf{B}^3[\mathbf{c}_i, 1] | i \in I\}$ *is an arbitrary packing of unit balls in* $\mathbb{E}^3$ *and* $\mathbf{V}_i$ *denotes the Voronoi cell assigned to* $\mathbf{B}^d[\mathbf{c}_i, 1]$, $i \in I$*, then* $\frac{\omega_3}{\mathrm{vol}_3\left(\mathbf{V}_i \cap \mathbf{B}^3[\mathbf{c}_i, \sqrt{2}]\right)} \leq \frac{\omega_3}{\mathrm{vol}_3(\mathbf{D})} < 0.7547 < \sigma_3 = 0.7797\ldots$*, where* $\mathbf{D}$ *stands for a regular dodacahedron of inradius* 1*. Hence,* $\hat{\delta}_3 < 0.7547$*. The latter inequality and (6) yield that if* $\mathcal{P}$ *is an arbitrary LS-packing of* $n > 1$ *unit balls in* $\mathbb{E}^3$*, then* $c(\mathcal{P}) \leq \left\lfloor 3n - \hat{\delta}_3^{-\frac{2}{3}} n^{\frac{2}{3}} \right\rfloor \leq \left\lfloor 3n - 1.206 n^{\frac{2}{3}} \right\rfloor$.

Remark 16. *Let* $\mathcal{P} := \{\mathbf{B}^d[\mathbf{c}_i, \frac{1}{2}] | \mathbf{c}_i \in \mathbb{Z}^d, 1 \leq i \leq n\}$ *be an arbitrary packing of* n *unit diameter balls with centers having integer coordinates in* $\mathbb{E}^d$. *Clearly,* $\mathcal{P}$ *is a TS-packing. Then let* $c_{\mathbb{Z}^d}(n)$ *denote the largest* $c(\mathcal{P})$ *for packings* $\mathcal{P}$ *of* n *unit diameter balls of* $\mathbb{E}^d$ *obtained in this way. It is proved in* [BSS16] *that*

$$dN^d - dN^{d-1} \leq c_{\mathbb{Z}^d}(n) \leq \left\lfloor dn - dn^{\frac{d-1}{d}} \right\rfloor \tag{7}$$

for $N \in \mathbb{Z}$ *satisfying* $0 \leq N \leq n^{\frac{1}{d}} < N+1$, *where* $d > 1$ *and* $n > 1$. *Note that if* $N = n^{\frac{1}{d}} \in \mathbb{Z}$, *then the lower and upper estimates of (7) are equal to* $c_{\mathbb{Z}^d}(n)$. *Furthermore,*

$$c_{\mathbb{Z}^2}(n) = \lfloor 2n - 2\sqrt{n} \rfloor \tag{8}$$

for all $n > 1$. *We note that* [Nev95] *(resp.,* [AC96]*) generates an algorithm that lists some (resp., all) packings* $\mathcal{P} = \{\mathbf{B}^d[\mathbf{c}_i, \frac{1}{2}] | \mathbf{c}_i \in \mathbb{Z}^d, 1 \leq i \leq n\}$ *with* $c(\mathcal{P}) = c_{\mathbb{Z}^d}(n)$ *for* $d \geq 4$ *(resp.,* $d = 2, 3$*) and* $n > 1$.

[BSS16] K. Bezdek, B. Szalkai, and I. Szalkai, *On contact numbers of totally separable unit sphere packings*, Discrete Math. **339** (2016), no. 2, 668–676. MR3431379

[Nev95] E. J. Neves, *A discrete variational problem related to Ising droplets at low temperatures*, J. Statist. Phys. **80** (1995), no. 1-2, 103–123. MR1340555

[AC96] L. Alonso and R. Cerf, *The three-dimensional polyominoes of minimal area*, Electron. J. Combin. **3** (1996), no. 1, Research Paper 27, approx. 39. MR1410882

Problem 9. *Let* $\mathcal{P}$ *be an arbitrary LS-packing (resp., TS-packing) of* $n > 1$ *unit diameter balls in* $\mathbb{E}^d$, $d \geq 3$. *Then prove or disprove that* $c(\mathcal{P}) \leq c_{\mathbb{Z}^d}(n)$.