Dense sphere packings in dimensions 8 and 24 - at the crossroad of Number Theory, Fourier analysis and Geometry

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Quadratic forms

Lagrange (Extremal values of quadratic forms)

For positive definite quadratic form Q on $\mathbb{R}^n$, find

$$\min\{Q(z) : z \in \mathbb{Z}^n \backslash \{o\}\}$$

$\exists\, \Phi \in \mathrm{GL}(n,\mathbb{R})$ s.t. $Q(x) = \|\Phi x\|^2 \Longrightarrow$ for $\Lambda = \Phi\mathbb{Z}^n$ (lattice)

$$\min\left\{\sqrt{Q(z)} : z \in \mathbb{Z}^n \backslash \{o\}\right\} = \min\{\|x\| : x \in \Lambda\backslash\{o\}\} = \lambda(\Lambda)$$

$\det\Lambda = |\det\Phi| = \sqrt{\det Q}$

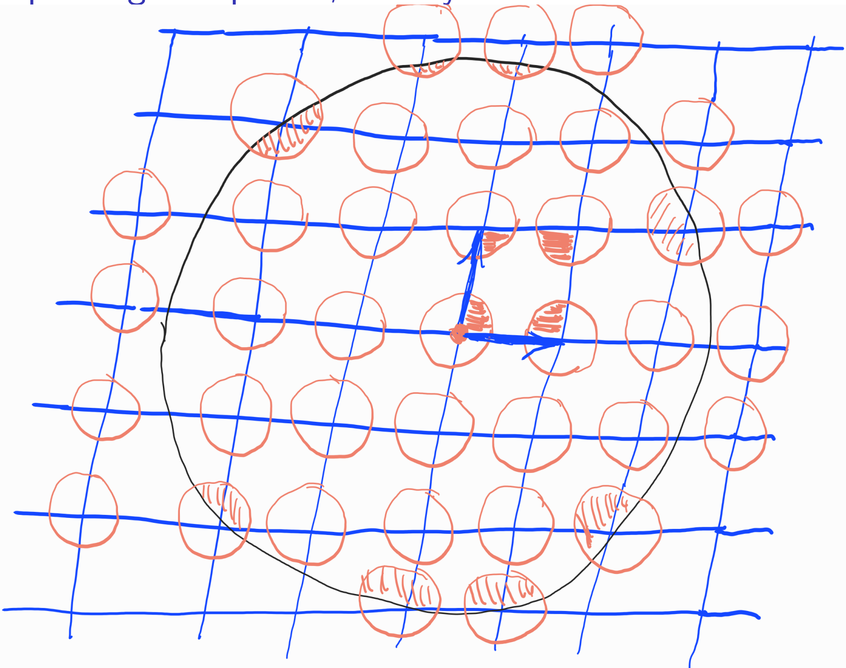
$\{e_i\}$ basis of $\mathbb{Z}^n \Longrightarrow \{w_i = \Phi e_i\}$ basis of Λ ($\Longleftrightarrow \Lambda = \sum_{i=1}^n \mathbb{Z}w_i$)

Problem

Find $\max\{\lambda(\Lambda) : \det\Lambda = \mathrm{constant}\} \Longleftrightarrow$

$$\max\left\{\frac{\lambda(\Lambda)^n}{\det\Lambda} : \Lambda \subset \mathbb{R}^n \text{ lattice}\right\}$$

Lattice packings of spheres, density



Lattice packings of spheres, Dirichlet-Voronoi cell, density

$\Lambda \subset \mathbb{R}^n$ lattice, $B^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$
For $r > 0$,

$\Lambda + rB^n$ packing $\Longleftrightarrow \mathrm{int}(x + rB^n) \cap \mathrm{int}(y + rB^n) = \emptyset$ for $x \neq y \in \Lambda$

$$\Longleftrightarrow \|x - y\| \geq 2r \text{ for } x \neq y \in \Lambda \Longleftrightarrow r \leq \frac{\lambda(\Lambda)}{2}$$

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$$\iff \|x - y\| \geq 2r \text{ for } x \neq y \in \Lambda \iff r \leq \frac{\lambda(\Lambda)}{2}$$

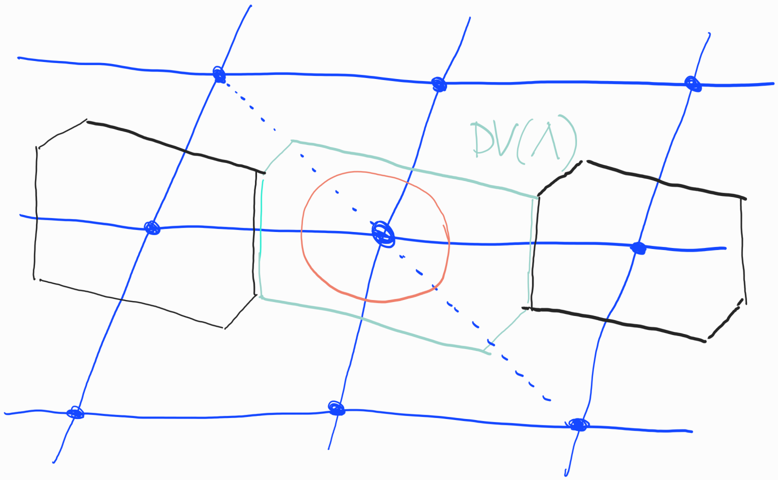
Dirichlet-Voronoi cell: $\mathrm{DV}(\Lambda) = \{x \in \mathbb{R}^n : \|x\| \leq \|x - z\| \; \forall z \in \Lambda\}$
$\Lambda + rB^n$ packing $\iff rB^n \subset \mathrm{DV}(\Lambda)$

Density of the packing

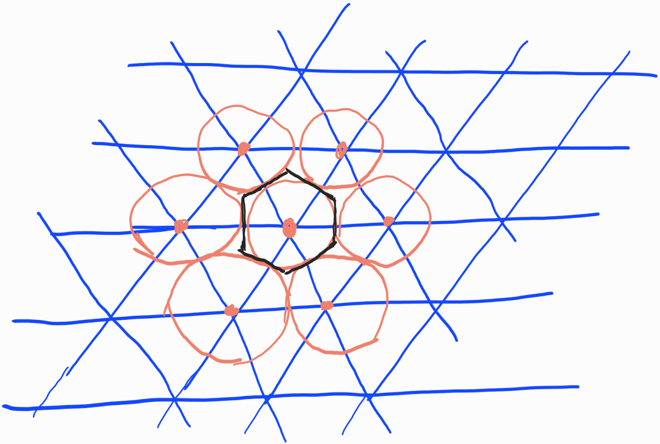
$$\delta(\Lambda + rB^n) = \frac{|rB^n|}{\det \Lambda} = \frac{|rB^n|}{|\mathrm{DV}(\Lambda)|} = \lim_{R \to \infty} \frac{\left|(\Lambda + rB^n) \cap RB^n\right|}{|RB^n|}$$

Given Λ, max density when $r = \frac{\lambda(\Lambda)}{2} \implies = \frac{\lambda(\Lambda)^n}{\det \Lambda} \cdot \frac{|B^n|}{2^n}$

Dirichlet-Voronoi cell, $|DV(\Lambda)| = \det \Lambda$



Hexagonal packing



Some lattices

$$D_n = \left\{(z_1,\ldots,z_n)\in\mathbb{Z}^n : \sum_{i=1}^n z_i \equiv 0 \mod 2\right\}$$
$$\Longrightarrow \lambda(D_n)=\sqrt{2} \text{ and } \det D_n = 2$$
$$E_8 = \left\{(z_1,\ldots,z_8)\in\mathbb{Z}^8\cup(\mathbb{Z}+\tfrac{1}{2})^8 : \sum_{i=1}^n z_i \equiv 0 \mod 2\right\}$$
$$\Longrightarrow \lambda(E_8)=\sqrt{2} \text{ and } \det E_8 = 1$$

Polar of a lattice Λ: $\Lambda^* = \{y\in\mathbb{R}^n : \langle y,z\rangle\in\mathbb{Z} \text{ for } z\in\Lambda\}$,
$\Longrightarrow \det\Lambda^* = 1/\det\Lambda$
E_8 unique unimodular even lattice in $\mathbb{R}^8$

- $\Lambda\subset\mathbb{R}^n$ unimodular if $\Lambda=\Lambda^*$ ($\Longrightarrow \det\Lambda = 1$)
- $\Lambda\subset\mathbb{R}^n$ even if $\|z\|^2\in 2\mathbb{Z}$ for $z\in\Lambda$

Leech lattice:
$\Lambda_{24}\subset\mathbb{R}^{24}$ is the unique even unimodular lattice in $\mathbb{R}^{24}$ with $\lambda(\Lambda_{24})=2$

Densest lattice packings

Newton (kissing) number: $N(B^n)$

$$\delta_L(B^n) = \max\{\delta(\Lambda + rB^n) : \Lambda \subset \mathbb{R}^n \text{ lattice and } \Lambda + rB^n \text{ packing}\}$$
$$= \max\left\{\frac{\lambda(\Lambda)^n}{\det \Lambda} \cdot \frac{|B^n|}{2^n} : \Lambda \subset \mathbb{R}^n \text{ lattice}\right\}$$

- n=2: A_2 (Lagrange, 1773), $N(B^2) = 6$, "tight"
- n=3: $A_3 = D_3$ (Gauss, 1831), $N(B^3) = 12$, "not tight"
- n=4,5: D_4, D_5 (Korkin, Zolotarev, 1872, 1877), $N(B^4) = 24$, "not tight"
- n=6,7,8: E_6, E_7, E_8 (Blichfeldt, 1929), $N(B^8) = 240$, "tight"
- n=24: Λ_{24} (Cohn, Kumar, 2004), $N(B^{24}) = 196560$, "tight"

General packings vs. Periodic packings

$\Gamma + rB^n$ is a packing for $\Gamma \subset \mathbb{R}^n$ if $\|x - y\| \geq 2r$ for $y \neq z \in \Gamma$

Upper density: $\delta_+(\Gamma + rB^n) = \limsup_{R\to\infty} \frac{|(\Gamma + rB^n)\cap RB^n|}{|RB^n|}$
Packing density: $\delta(B^n) = \sup\{\delta_+(\Gamma + rB^n) : \Gamma + rB^n \text{ packing}\}$

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$\Gamma + rB^n$ is a periodic packing

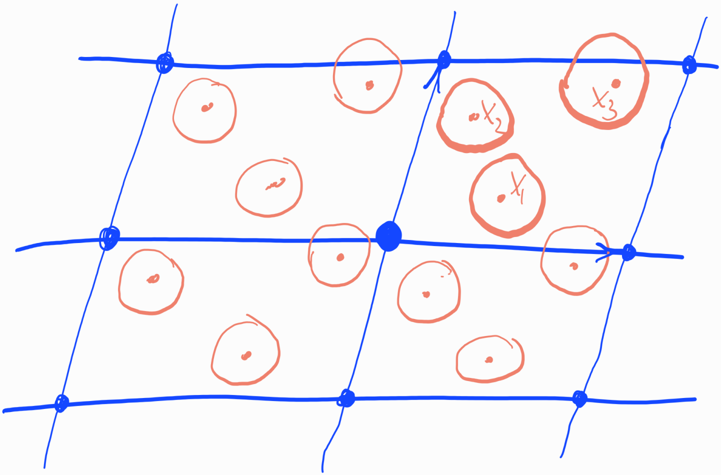
- $\iff \exists \Lambda \subset \mathbb{R}^n$ lattice s.t. $\Gamma + \Lambda = \Gamma$ (invariant)
- $\iff \Gamma = \Lambda + \{x_1, \ldots, x_m\}$ where $x_i - x_j \notin \Lambda$ for $i \neq j$

Density of this periodic packing:

$$\delta(\Gamma + rB^n) = \frac{m \cdot |rB^n|}{\det \Lambda} = \lim_{R\to\infty} \frac{|(\Gamma + rB^n) \cap RB^n|}{|RB^n|}$$

Groemer:
$\delta(B^n) = \sup\{\delta(\Gamma + rB^n) : \Gamma + rB^n \text{ periodic packing}\}$

Periodic packings



Known packing densities

- n=2: $\delta(B^2) = \delta(A_2 + B^2)$ (Thue, 1890, László Fejes Tóth, 1942)
- n=3 (Kepler conjecture): $\delta(B^3) = \delta(D_3 + \frac{\sqrt{2}}{2} B^3)$ (Hales, 2010)

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- n=8: $\delta(B^8) = \delta(E_8 + \frac{\sqrt{2}}{2} B^8)$ (Viazovska, 2017)
- n=24: $\delta(B^{24}) = \delta(\Lambda_{24} + B^{24})$ (Cohn, Kumar, Miller, Radchenko, Viazovska, 2017)

From Fourier transform to Modular forms

$\hat{f}(y) = \int_{\mathbb{R}^n} f(x) e^{2\pi i \langle x,y \rangle} \, dx$

Lemma (Cohn, Elkies) $r > 0$, $f \not\equiv 0$ Schwartz function on $\mathbb{R}^n$

(i) $f(x) \le 0$ if $\|x\| \ge 2r$;

(ii) $\hat{f}(y) \ge 0$ if $y \in \mathbb{R}^n$.

Then $\delta(B^n) \le \dfrac{f(o)}{\hat{f}(o)} \cdot |rB^n|$.

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Viazovska: $\exists \, g_8 : \mathbb{R} \to \mathbb{R}$ such that for $f_8(x) = g_8(\|x\|^2)$ and $r = \frac{\sqrt{2}}{2}$, f_8 satisfies (i) and (ii) and

$$\frac{f_8(o)}{\widehat{f_8}(o)} \cdot |rB^8| = \delta\left(E_8 + r\,B^8\right)$$

Viazovska et al: $\exists \, g_{24} : \mathbb{R} \to \mathbb{R}$ such that for $f_{24}(x) = g_{24}(\|x\|^2)$,

$$\frac{f_{24}(o)}{\widehat{f_{24}}(o)} \cdot |B^{24}| = \delta\left(\Lambda_{24} + B^{24}\right)$$

Proof of the Lemma about Fourier transform

Equivalent

$\Gamma + rB^n$ periodic packing, $\Gamma = \Lambda + \{x_1, \ldots, x_m\}$ where $x_i - x_j \notin \Lambda$ for $i \neq j$ and $\|x - y\| \geq 2r$ for $y \neq z \in \Gamma \Longrightarrow \frac{m \cdot |rB^n|}{\det \Lambda} \leq \frac{f(o)}{\hat{f}(o)} \cdot |rB^n|$

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Poisson for $v \in \mathbb{R}^n$:

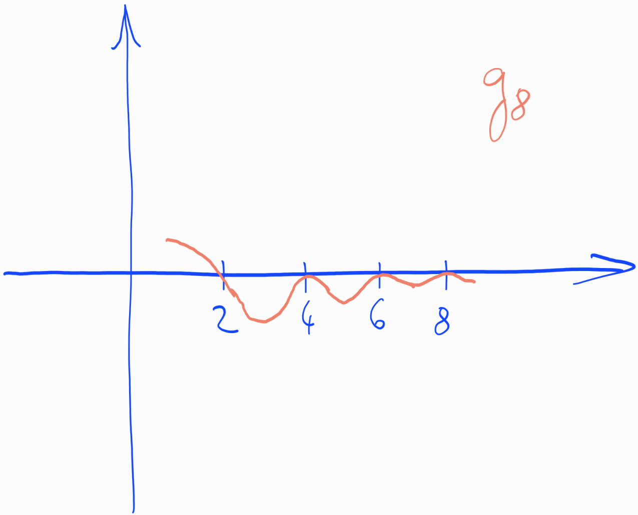
$$\sum_{x \in \Lambda} f(x + v) = \frac{1}{\det \Lambda} \sum_{y \in \Lambda^*} e^{-2\pi i \langle v, y \rangle} \hat{f}(y).$$

$$m \cdot f(o) \geq \sum_{x \in \Lambda,\ 1 \leq j,k \leq m} f(x + x_j - x_k)$$

$$= \frac{1}{\det \Lambda} \sum_{y \in \Lambda^*} \hat{f}(y) \sum_{1 \leq j,k \leq m} e^{-2\pi i \langle x_j - x_k, y \rangle}$$

$$= \frac{1}{\det \Lambda} \sum_{y \in \Lambda^*} \hat{f}(y) \left| \sum_{1 \leq j \leq m} e^{-2\pi i \langle x_j, y \rangle} \right|^2 \geq \frac{m^2}{\det \Lambda} \cdot \hat{f}(o).$$

Maryna Viazovska's function g_8



Stability versions by B., Radchenko, Ramos (Crelle)

Theorem $\Lambda \subset \mathbb{R}^8$ lattice, $\lambda(\Lambda) \geq \sqrt{2}$ and $\det \Lambda \leq 1 + \varepsilon \implies \exists$ basis $w_1, \dots, w_8$ of Λ and basis $u_1, \dots, u_8$ of E_8 such that

- $\forall \, \|w_i\| \leq 2^{16}$,
- $\forall \, \|w_i - u_i\| \leq c_8 \varepsilon$.

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Theorem $R_\varepsilon = \frac{|\log \varepsilon|}{\log |\log \varepsilon|}$, $R_\varepsilon B^8 \subset W \subset R_\varepsilon^{20} B^8$ and $\Gamma + \frac{\sqrt{2}}{2} B^8$ periodic packing with $\delta(\Gamma + \frac{\sqrt{2}}{2} B^8) \geq (1 - \varepsilon)\delta(E_8 + \frac{\sqrt{2}}{2} B^8) \Longrightarrow$ With probability at least $1 - \frac{c_8}{\sqrt{R_\varepsilon}}$, an $x \in \mathbb{R}^8$ satisfies

- $\#(\Gamma \cap (x + W)) \geq \left(1 - \frac{c_8}{\sqrt{R_\varepsilon}}\right) \cdot \#(E_8 \cap (x + W))$,
- $d_H(\Phi Z, \Gamma \cap (x + W)) \leq \frac{c_8}{\sqrt{R_\varepsilon}}$ for a $Z \subset E_8$ and isometry Φ.

For compact $X, Y \subset \mathbb{R}^8$,

$$d_H(X, Y) = \min\left\{\varrho \geq 0 : \; X \subset Y + \varrho B^8 \text{ and } Y \subset X + \varrho B^8\right\}$$