Exact covering with unit disks

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Thursday, July 4, 2024
(with minor corrections and improvements)

Naoki Inaba's puzzles

- Naoki Inaba is a Japanese puzzle creator who has developed over 400 different puzzles. One of them is about covering sets of points with disks.

Problem

Show that any set of 10 points in $\mathbb{R}^2$ can be covered by nonoverlapping unit disks.

- Inaba elegantly solved this problem with a probabilistic method.
- We will present an area-version of this proof on the next two slides.

A solution to Inaba's disk covering problem

Definition

Let σ_2 be the largest $n \in \mathbb{N}$ such that any set of n points in $\mathbb{R}^2$ can be covered by disjoint unit disks.

Theorem

$\sigma_2 \geq 10$.

Proof.

Let $X = \left\{\mathbf{x}^1, \ldots, \mathbf{x}^n\right\}$. Consider the hexagonal lattice A_2, scaled so that the minimum distance between distinct points is 2. This packing has a density of $\delta\left(A_2\right) = \frac{\pi}{2\sqrt{3}} \approx 0.9069$.

Let $\mathscr{A}_2 := \left\{\mathbf{c} + B^2 \subset \mathbb{R}^2 \,\middle|\, \mathbf{c} \in A_2\right\}$, where $B^2 := \left\{\mathbf{z} \in \mathbb{R}^2 \,\middle|\, \|\mathbf{z}\| < 1\right\}$ is the unit open disk. We wish to find a translation vector $\mathbf{t} \in \mathbb{R}^2$ such that the translated collection $\mathbf{t} + \mathscr{A}_2 := \left\{\mathbf{t} + D \subset \mathbb{R}^2 \,\middle|\, D \in \mathscr{A}_2\right\}$ of disks covers X.

A solution to Inaba's disk covering problem

Proof (continued).

Each $\mathbf{x}^i \in X$ has a corresponding "forbidden" set

$$F_{\mathbf{x}^i} := \left\{ \mathbf{t} \in \mathbb{R}^2 \,\middle|\, \mathbf{x}^i \notin D \text{ for all } D \in \mathbf{t} + \mathscr{A}_2 \right\}$$

of translation vectors $\mathbf{t}$ such that $\mathbf{t} + \mathscr{A}_2$ does not cover $\mathbf{x}^i$. If

$$\bigcup_{i=1}^{n} F_{\mathbf{x}^i} \neq \mathbb{R}^2$$

then there exists a translation vector $\mathbf{t}' \in \mathbb{R}^2$ that is not in any forbidden set. So X is covered by $\mathbf{t}' + \mathscr{A}_2$.

Each $F_{\mathbf{x}^i}$ takes up $1 - \delta(A_2) \approx 0.0931 < \frac{1}{10}$ of the plane, so the union cannot be all of the plane. □

Probabilistic/area lower bound for σ_2

- Let Λ be any lattice in $\mathbb{R}^2$ and let $\delta(\Lambda)$ be its density. We have

$$\sigma_2 \geq \left\lceil \frac{1}{1-\delta(\Lambda)} \right\rceil - 1.$$

Lower and upper bounds for σ_2

- We have $\sigma_2 < \infty$.
  - A point set X that extends beyond one disk and consists of closely spaced points cannot be covered by disjoint disks.
  - Intuitively, this case is similar to the problem of covering $\operatorname{conv} X$ by disjoint disks.
- Currently we know that $12 \leq \sigma_2 \leq 44$.
  - Aloupis, Hearn, Iwasawa, and Uehara (2012) achieved $\sigma_2 \geq 12$ using a more elaborate probabilistic strategy on vertical line subsets of the plane.
  - The upper bound was reduced from $\sigma_2 < 60$ (Winkler 2010) to $\sigma_2 < 55$ (Elser 2011) and $\sigma_2 < 53$ (Okayama, Kiyomi, and Uehara 2012).
  - Most recently, Aloupis, Hearn, Iwasawa, and Uehara (2012) found a 50-point subset of A_2 and a 45-point set consisting of three concentric circles (using a computer) that cannot be covered by disjoint sets.

The exact cover relaxation

- Our work focuses on a relaxed version of Inaba's problem where the disks are allowed to overlap, but each point is covered by only one disk: **exact covering**.

Definition

Let $\widehat{\sigma}_2$ be the largest $n \in \mathbb{N}$ such that any set of n points in $\mathbb{R}^2$ can be exactly covered.

- Every disjoint cover is an exact cover.
- $\widehat{\sigma}_2$ is finite for the same reason that σ_2 is finite.
- Hence we have the basic inequalities

$$\sigma_2 \leq \widehat{\sigma}_2 < \infty.$$

A lower bound for $\widehat{\sigma}_2$

The Extension Argument

Definition

We say that a point $\mathbf{x} \in X$ is a **boundary point** of X if $\mathbf{x}$ is on the boundary of $\operatorname{conv} X$.

- László Kozma (personal communication) found a simple proof of the inequality $\widehat{\sigma}_2 \geq \sigma_2 + 3$.

Lemma (Extension Argument)

Let k be the number of boundary points of $\operatorname{conv} X$.

1. *If $|X| \leq \sigma_2 + k$ then X can be exactly covered. ($|X|$ = cardinality)*
2. *If $k \leq 2$ then X can be exactly covered regardless of $|X|$.*
3. $\widehat{\sigma}_2 \geq \sigma_2 + 3$.

Proof of the Extension Argument

Proof of Part 1 of the Extension Argument.

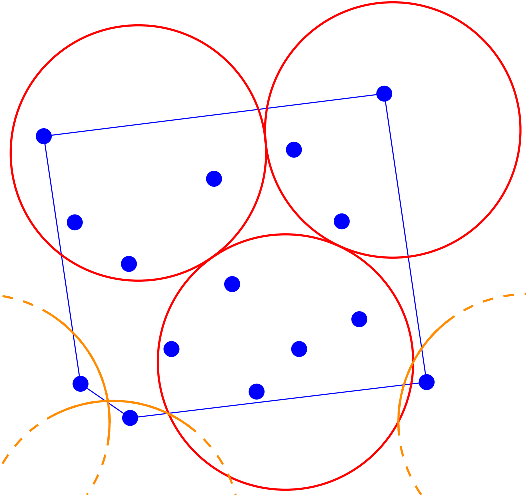
Let $X \in \mathscr{X}$. Cover all the non-boundary points of X by a collection $\mathscr{D}'$ of disjoint disks. For each boundary point $\mathbf{b} \in X$:

1. If $\mathbf{b}$ is already covered by $\mathscr{D}'$, then do nothing.
2. If $\mathbf{b}$ is not covered by $\mathscr{D}'$, then add a new disk that covers $\mathbf{b}$ but no other point of X.
   Such a disk exists because $\operatorname{conv} X$ is convex and B^2 is strictly convex.

The resulting collection $\mathscr{D} := \mathscr{D}' \cup \{\text{all new disks from case 2}\}$ is an exact cover of X. □

- The Extension Argument with Aloupis, Hearn, Iwasawa, and Uehara's result of $\sigma_2 \geq 12$ implies $\widehat{\sigma}_2 \geq 15$.

Proof of the Extension Argument



Generalizing the Extension Argument

- The Extension Argument is limited by the case where $\operatorname{conv} X$ is a triangle, because then X may have only three boundary points.
- If we can loosen the boundary point condition so that every X has at least four "generalized boundary points," then we have $\widehat{\sigma}_2 \geq 16$.

Definition

We say that a point $\mathbf{b} \in X$ is a **generalized boundary point** of X if there exists a disk $\mathbf{c} + B^2$ that contains $\mathbf{b}$ but no other point of X.

Lemma (Generalized Extension Argument)

Let k be the number of generalized boundary points of $\operatorname{conv} X$.

1. *If $|X| \leq \sigma_2 + k$ then X can be exactly covered.*
2. *If $k \leq 3$ then X can be exactly covered regardless of $|X|$.*
3. $\widehat{\sigma}_2 \geq \sigma_2 + 4$.

The lower bound $\widehat{\sigma}_2 \geq 17$

- Suppose that $T := \operatorname{conv} X$ is a triangle. Let R_T be its circumradius.
- We focus on triangles since they give the "worst-case scenario" of three boundary points.
- The proof of the Generalized Extension Argument involves four different cases, listed below from "largest" T to "smallest" T:
  1. At least one side of T has length ≥ 2.
  2. All sides of T have length < 2 but $R_T > 1$.
  3. All sides of T have length < 2 and $R_T = 1$.
  4. All sides of T have length < 2 and $R_T < 1$.
- For cases 1 and 2, T is "large enough" for X to have at least four generalized boundary points.
- For cases 3 and 4, T is "small enough" for X to be exactly covered regardless of the number of points.
- Then $\widehat{\sigma}_2 \geq \sigma_2 + 4 \geq 16$.
- We reach $\widehat{\sigma}_2 \geq 17$ using complicated technical arguments.

An upper bound

Nets and blockers

- We obtain an upper bound for $\widehat{\sigma}_2$ by constructing a point set which cannot be exactly covered.
- Let X be a nonempty subset of $\mathbb{R}^2$, $\mathbf{y} \in \mathbb{R}^2$, and $\varepsilon > 0$. The distance from $\mathbf{y}$ to X is $\operatorname{dist}(\mathbf{y}, X) := \inf \{\|\mathbf{y} - \mathbf{x}\| \mid \mathbf{x} \in X\}$ and the **ε-extension** of X is

$$X_\varepsilon := \left\{\mathbf{y} \in \mathbb{R}^2 \,\middle|\, \operatorname{dist}(\mathbf{y}, X) \leq \varepsilon\right\}.$$

- The set X is an **ε-net** of $M \subseteq \mathbb{R}^2$ if $M \subseteq X_\varepsilon$.
- The set M is an **ε-blocker** if every point set X that is an ε-net of M cannot be exactly covered.
- The minimal cardinality of an ε-net of M is called the **covering number** of M and is denoted by $N(M, \varepsilon)$.
- We have $\widehat{\sigma}_2 < N(M, \varepsilon)$ for any ε-blocker M.

The upper bound $\widehat{\sigma}_2 < 657$

- So we look for a suitable M and ε.

Proposition

Let $\varepsilon \in \left(0, 7 - \sqrt{48} \approx 0.0718\right]$ and $r \geq \frac{3}{2}(1+\varepsilon) \approx 1.608$. The disk $M = rB^2$ is an ε-blocker.

- It follows that for any $\mathbf{y} \in \mathbb{R}^2$,

$$X(\mathbf{y}) := \left(\mathbf{y} + \frac{\varepsilon\sqrt{3}}{2} A_2\right) \cap (r+\varepsilon) B^2.$$

is an ε-net of rB^2.

- So we search for an appropriate vector $\mathbf{y}$ that minimizes $|X(\mathbf{y})|$. The best result that we found is

$$\widehat{\sigma}_2 < \left| X\left(\begin{pmatrix} 0.035 \\ -0.055 \end{pmatrix}\right)\right| = 657.$$

Higher dimensions

Higher dimensions are not very interesting

- Inaba's problem has a straightforward generalization to any $\mathbb{R}^d$.

Definition

For any $d \in \mathbb{N}$, let σ_d and $\widehat{\sigma}_d$ be the largest $n \in \mathbb{N}$ such that any set of n points in $\mathbb{R}^d$ can be covered by disjoint unit disks or can be exactly covered, respectively.

- Many of our methods do not easily scale to higher dimensions.
- Those that do tend to have limited effectiveness as d increases.

Lower bounds for σ_d

- We can show that $\sigma_d \geq 3$ for all $d \in \mathbb{N}$ using a similar approach to our proof of the Generalized Extension Argument.
- The density of the densest infinite packing is ≤ 0.5 for all $d \geq 5$, so the probabilistic/area method reduces to a triviality.

Dimension	Densest known packing	Density	Lower bound $\sigma_d \geq \left\lceil \frac{1}{1-\delta(\Lambda)} \right\rceil - 1$
2	A_2	$\frac{\pi}{2\sqrt{3}} \approx 0.9069$	10
3	D_3	$\frac{\pi}{3\sqrt{2}} \approx 0.7405$	3
4	D_4	$\frac{\pi^2}{16} \approx 0.6169$	2
5	D_5	$\frac{\pi^2}{15\sqrt{2}} \approx 0.4653$	1
6	E_6	$\frac{\pi^3}{48\sqrt{3}} \approx 0.3729$	1

A simple lower bound for $\widehat{\sigma}_d$

- In $\mathbb{R}^d$, the Extension Argument is limited by a d-dimensional simplex instead of a triangle.

Lemma (d-dimensional Extension Argument)

Let k be the number of generalized boundary points of $\operatorname{conv} X \subset \mathbb{R}^d$.

1. *If $|X| \leq \sigma_2 + k$ then X can be exactly covered.*
2. *If $k \leq d+1$ then X can be exactly covered regardless of $|X|$.*
3. $\widehat{\sigma}_d \geq \sigma_d + (d+2)$.

- Hence we obtain, in general,

$$\widehat{\sigma}_d \geq 3 + (d+2) = d+5.$$

- We also reach $\widehat{\sigma}_3 \geq 9$ using some complicated arguments.

Possible future directions

Future directions

- We have a preprint, which provides the technical details and additional figures:

  https://arxiv.org/abs/2401.15821

- Other convex bodies.
- Computational complexity.

Thank you for your attention!