Extending Embeddings of Simplicial Complexes and Linking Gadgets

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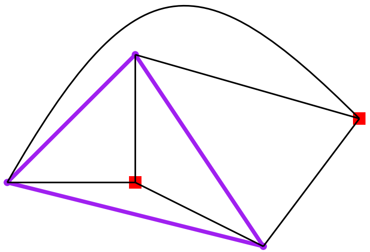
Discrete Geometry Days^3

Linking Gadgets - Introduction

Consider the following graph G that contains two *disjoint* spheres S^1 and S^0



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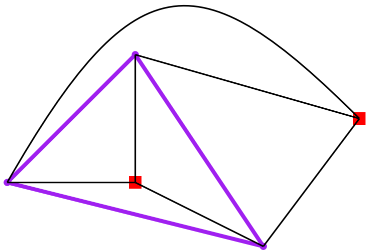

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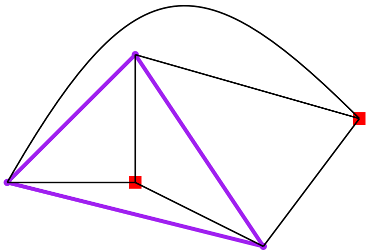

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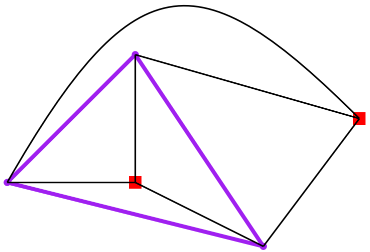

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- Let $f\colon G \to \mathbb{R}^2$ be an almost embedding (= neighbouring edges can intersect). What is the linking degree between the spheres? Fulek, Garaev, Kynčl

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Is there a *simplicial complex* called *linking gadget* $G(k, \ell)$ that contains two *disjoint* spheres S^k and S^ℓ such that

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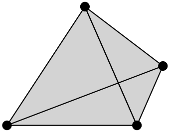
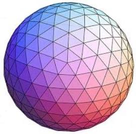
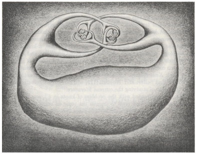


Linear embedding



PL embedding



Topological embedding

Linking gadgets in higher dimensions - "results"

Embeddings - trivial fact

One can find $G(k,\ell)$, $k > \ell$ with $\dim G(k,\ell) = (k+\ell)$.

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Segal, Spiez, F, Karasev, Skopenkov

There exists a simplicial complex $GX(k, \ell)$ with $\dim GX(k, \ell) = k > \ell$, where for any almost-embedding, the spheres are linked with any odd linking number

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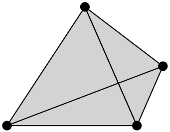
Open question(s) (Karasev, Skopenkov)

Any simplicial complex $GX(k,\ell)$, $\dim GX(k,\ell) = k > \ell$, that admits an (almost)-embedding $f\colon GX(k,\ell) \to \mathbb{R}^{k+\ell+1}$ with linking degree ± 1, admits an (almost)-embedding $g\colon GX(k,\ell) \to \mathbb{R}^{k+\ell+1}$ with linking degree $2m+1, m \in \mathbb{Z}$.

Why?

i.e. the motivation

Embeddings of simplicial complexes



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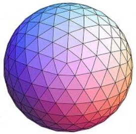

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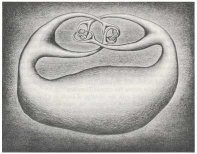

Topological embedding

$\mathrm{EMBED}_{k\to d}$

Input: $d \geq 2$, K finite simplicial complex, $\dim K = k$.

Output: $K \hookrightarrow_{\mathrm{PL}} \mathbb{R}^d$: YES / NO .

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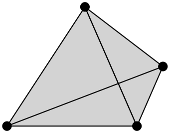

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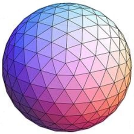

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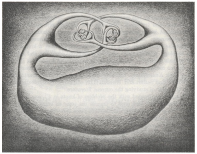

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$\mathrm{EXTEMBED}_{k \to d}$

Input: $d \geq 2$, K finite simplicial complex, $\dim K = k$, $L \subseteq K$, $f' \colon L \hookrightarrow_{\mathrm{PL}} \mathbb{R}^d$.
Output: $f \colon K \hookrightarrow_{\mathrm{PL}} \mathbb{R}^d$, $f|L = f'$: YES / NO .

$\mathrm{EXTEMBED}_{k \to d}$ - Status Quo

$k \setminus d$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	L																		
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$\mathrm{EXTEMBED}_{k\to d}$ - Metastable range

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1	L																		
2	L																		
3					P														
4							P												
5								P	P										
6										P	P								
7											P	P	P						
8													P	P	P				
9														P	P	P	P		
10																P	P	P	P

$k \leq \frac{2d-3}{3}$

Čadek, Krčál, Vokřínek (2017) : decidable in polynomial time

$\mathrm{EXTEMBED}_{k \to d}$ - Undecidability I

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- Matoušek, Tancer, Wagner (2011) $\mathrm{EMBED}_{k\to d}$, $d \geq 4$, is NP-hard outside the metastable range.
- de Mesmay, Rieck, Sedgwick, Tancer (2018) $\mathrm{EMBED}_{2\to 3}$, $\mathrm{EMBED}_{3\to 3}$ are NP-hard.
- Matoušek, Sedgwick, Tancer, Wagner (2018) $\mathrm{EMBED}_{2\to 3}$, $\mathrm{EMBED}_{3\to 3}$ are decidable.

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4				U		U	P												
5				U	U	U	U	P	P										
6					U	U	U	U	U	P	P								
7						U	U	U	U	U	P	P	P						
8							U	U	U	U	U	U	P	P	P				
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- F, Wagner, Zhechev (2020): $\mathrm{EXTEMBED}_{k\to d}$ is undecidable for $8 \leq d < \lfloor \frac{3(k+1)}{2} \rfloor$.

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- Avvakumov, F, Wagner, Zhechev (2024?): $\mathrm{EXTEMBED}_{k\to d}$ is undecidable for $7 \le d \le \lfloor \frac{3(k+1)}{2} \rfloor$.

Systems of Diophantine equations

Hilbert's tenth problem

Input: A system of Diophantine equations.
Output: YES there is an integer solution ; NO otherwise.
Undecidable (Matiyasevich 1970; Robinson, Davis and Putnam).

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Q-SYM systems

$$\sum_{1 \leq i < j \leq r} a_{ij}^{(q)} x_i x_j = b^{(q)}, \quad q = 1, 2, \ldots, s, \ a_{ij}^{(q)}, b^{(q)} \in \mathbb{Z}$$

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$$\sum_{1 \leq i < j \leq r} a_{ij}^{(q)} (x_i y_j - x_j y_i) = b^{(q)}, \quad q = 1, 2, \ldots, s, \ a_{ij}^{(q)}, b^{(q)} \in \mathbb{Z}$$

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The extension problem is undecidable

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For $m \geq 2$, even the following is algorithmically undecidable.

We use Q-SYM:

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Extension problem rewritten

Original

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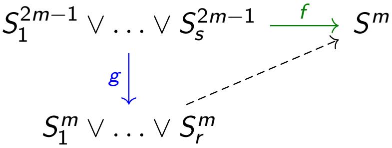

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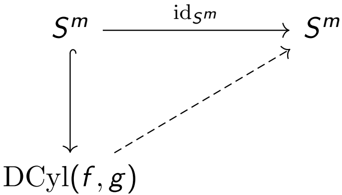

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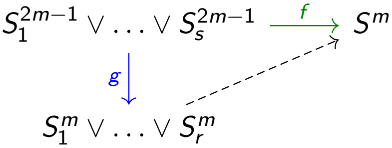

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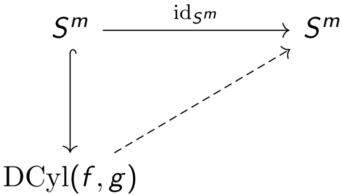

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Restated problem:

$$\begin{array}{ccc} S^m & \xrightarrow{\mathrm{id}_{S^m}} & S^m \\ \downarrow & \nearrow & \\ \mathrm{DCyl}(f,g) & & \end{array}$$

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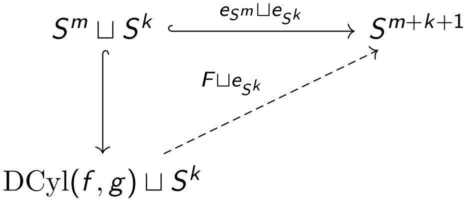
FACT: if k is large enough, the space $\mathrm{DCyl}(f,g)$ is embeddable in S^{m+k+1} by general position.

Extension problem rewritten again

$$
\begin{array}{ccc}
S^m \sqcup S^k & \xhookrightarrow{e_{S^m} \sqcup e_{S^k}} & S^{m+k+1} \\
\big\downarrow & \nearrow_{F \sqcup e_{S^k}} & \\
\mathrm{DCyl}(f, g) \sqcup S^k & &
\end{array}
$$

Extension problem rewritten again



F, Wagner, Zhechev : if $k \geq 2m + 1$, there exists an embedding $H \sqcup e_{S^k} : \mathrm{DCyl}(f, g) \sqcup S^k \to S^{m+k+1}$ such that $H \sim F$.

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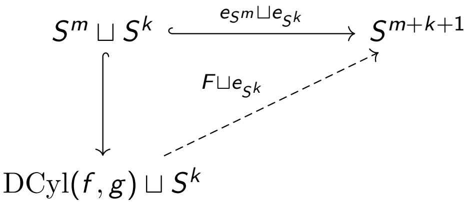

F, Wagner, Zhechev : if $k \geq 2m + 1$, there exists an embedding $H \sqcup e_{S^k} : \mathrm{DCyl}(f, g) \sqcup S^k \to S^{m+k+1}$ such that $H \sim F$.

Avvakumov, F, Wagner, Zhechev : $\mathrm{EXTEMBED}_{k \to d}$ is undecidable for $7 \leq d \leq \lfloor \frac{3(k+1)}{2} \rfloor$. (Choose $K = \mathrm{DCyl}(f, g) \sqcup S^k$, $L = S^m \sqcup S^k$, $f = e_{S^m} \sqcup e_{S^k}$)

$\mathrm{EXTEMBED}_{k \to d}$ - current state

$k \setminus d$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	L																		
2	L	DNPh																	
3		DNPh			P														
4				U		U	P												
5				U	U	U	U	P	P										
6					U	U	U	U	U	P	P								
7						U	U	U	U	U	P	P	P						
8							U	U	U	U	U	U	P	P	P				
9								U	U	U	U	U	U	P	P	P	P		
10									U	U	U	U	U	U	U	P	P	P	P

$\mathrm{EMBED}_{k \to d}$ - current state

$k \setminus d$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	L																		
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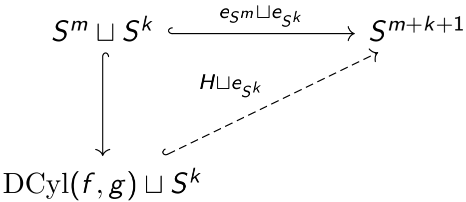
3		DNPh			P														
4				U			P												
5				U	U			P	P										
6					U	U				P	P								
7						U	U				P	P	P						
8							U	U					P	P	P				
9								U	U					P	P	P	P		
10									U	U						P	P	P	P

Undecidability of $\mathrm{EMBED}_{k\to d}$ - a theory

$\mathrm{EXTEMBED}_{2m+1\to 3m+2}$:



$\mathrm{EMBED}_{2m+1\to 3m+2}$:

Given a gadget that fixes the linking degree ± 1, we get

Undecidability of $\mathrm{EMBED}_{k\to d}$ - a theory

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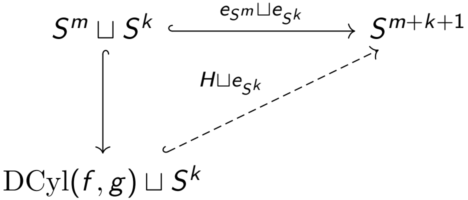

$\mathrm{EMBED}_{2m+1\to 3m+2}$:

Given a gadget that fixes the linking degree ± 1, we get

$$\mathrm{DCyl}(f,g) \cup_{S^m \sqcup S^k} G(k,m) \hookrightarrow S^{m+k+1}$$