Non-diagonal critical central sections of the cube

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Joint work with Gergely Ambrus

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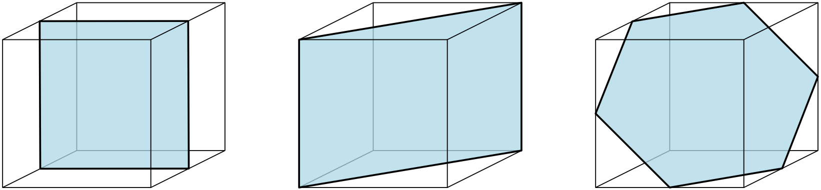
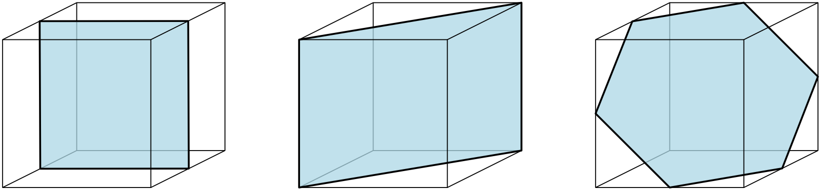
Introduction





Volume of central hyperplane sections of $Q_n = \left[-\frac{1}{2}, \frac{1}{2}\right]^n$ as a function of its normal vector:

$$\sigma(\mathrm{v}) = \mathrm{Vol}_{n-1}(Q_n \cap \mathrm{v}^\perp)$$

$\sigma(\mathrm{v})$ is invariant under scalings of v by a non-zero factor, and by embeddings in higher dimensions.

Previous results

- $\sigma(\mathrm{v})$ is calculable with Pólya's (1913) integral formula.
- Minimal sections are parallel to a facet of Q_n, their volume is 1 (Hadwiger (1972)).
- Maximal sections are orthogonal to the diagonal of a 2-dimensional face of Q_n, their volume is $\sqrt{2}$ (Ball (1986)).
- Noncentral and lower dimensional sections. (Ball, Ivanov, König, Moody, Stone, Vaaler, Zach, Zvavitch)
- Sections of the regular simplex and ℓ_p unit balls. (Chasapis, Dirksen, Meyer, Nayar, Pajor, Tkocz, Webb)

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Definition

$\mathrm{v} \in S^{n-1}$ is a critical direction if it is a critical point of the function $\sigma(\mathrm{v})$ on S^{n-1}. Then $Q_n \cap \mathrm{v}^\perp$ is a critical section.
Locally extremal sections are defined similarly.

Diagonal directions

Definition

Unit vectors parallel to the diagonal of a k-dimensional face of Q_n are called k-diagonal directions. Corresponding sections are k-diagonal sections.

Up to permutation of coordinates and change of signs, they have the form of

$$\mathrm{d}_{n,k} := \frac{1}{\sqrt{k}} (\underbrace{1, \ldots\ldots 1,}_{k} \ \underbrace{0, \ldots\ldots 0}_{n-k}).$$

- Based on Hensley's (1979) asymptotic formula, if $k \approx n$ then

$$\sigma(\mathrm{d}_{n,k}) \approx \sqrt{\frac{6}{\pi}}.$$

- Bartha, Fodor and González Merino (2020) showed that for fixed n the sequence of k-diagonal sections is strictly monotone increasing for $k \geq 3$.

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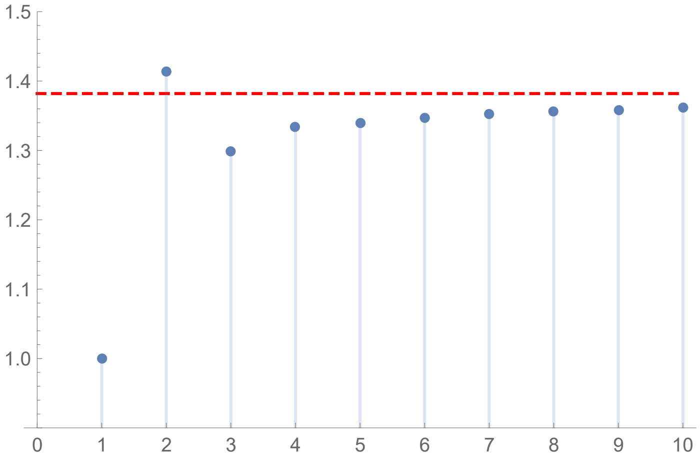

Maximality of diagonal sections

Theorem (Pournin (2023))

For all $k \geq 3$ k-diagonal sections are strictly locally maximal among central sections of Q_n for each $n \geq 4$.

We provided an alternative proof in the special case $k = n$.

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Theorem

The main diagonal section $Q_n \cap 1_n^{\perp}$ has strictly locally maximal volume among central sections of Q_n for each $n \geq 4$.

Our proof uses Lagrange multiplier methods. This requires first to show that diagonal directions are critical.

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Statement (Ambrus (2022))

$\mathrm{v} \in S^{n-1}$ *is a critical direction if and only if up to permuting coordinates and changing signs* $\mathrm{v} = \mathrm{e}_1$, *or*

$$\sigma(\mathrm{v}) = \frac{1}{\pi(1 - v_j^2)} \int_{-\infty}^{\infty} \prod_{i \neq j} \operatorname{sinc}(v_i t) \cdot \cos(v_j t)\, \mathrm{d}t$$

holds for each $j = 1, \ldots, n$.

Main idea of the proof

Suppose that $\mathsf{v} \in S^{n-1}$ is a critical direction. Then v is a stationary point of the Lagrange function

$$\Lambda(\mathsf{v}) = \sigma(\mathsf{v}) + \frac{\sigma(\mathsf{v})}{2} \cdot \left(|\mathsf{v}|^2 - 1\right)$$

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Its bordered Hessian matrix is

$$H(\Lambda(\mathrm{v})) = \begin{bmatrix} 0 & 2v_1 & 2v_2 & \dots & 2v_n \\ 2v_1 & & & & \\ 2v_2 & \multicolumn{4}{c}{\dfrac{\partial^2 \sigma}{\partial v_j \partial v_k}(\mathrm{v}) + \sigma(\mathrm{v}) \cdot \begin{cases} 0, & \text{if } j \neq k \\ 1, & \text{if } j = k \end{cases}} \\ \vdots & & & & \\ 2v_n & & & & \end{bmatrix}$$

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The main diagonal direction $\mathrm{d}_n = \frac{1}{\sqrt{n}} 1_n$ is critical. Denote with H_m the mth principal minors of $H(\Lambda(\mathrm{d}_n))$ $(m = 3, \dots, n)$.

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$$\forall m \quad (-1)^{m-1} H_m > 0$$

then $\sigma(\mathrm{v})$ is strictly locally maximal at d_n respect to $\mathrm{v} \in S^{n-1}$.

Further characterization of critical directions

Theorem (Ambrus (2022))

$\mathrm{v} = (v_1, \ldots, v_n) \in S^{n-1}$ is critical direction if and only if up to permuting coordinates and change of signs $\mathrm{v} = \mathrm{d}_{n,2}$, or there exists some $\mu > 0$ for which

$$\mathrm{Vol}_{n-1}(\mathrm{conv}(0 \cup (R_k \cap \mathrm{v}^\perp))) = \mu(1 - v_k^2)$$

holds true for each $k = 1, \ldots, n$, where
$R_k = \{(q_1, \ldots, q_n) \in Q_n : \ q_k = 1\}$.

Corollary (Ambrus (2022))

For $n = 2, 3$, all critical directions are diagonal. If $n = 4$, then the critical directions are either diagonal or parallel to the vector $(1, 1, 2, 2)$ up to permuting coordinates and changing signs.

Non-diagonal critical sections

Theorem

For all $n \geq 4$ there exist non-diagonal critical central sections of Q_n whose normal vector is not parallel to any of the coordinate axes.

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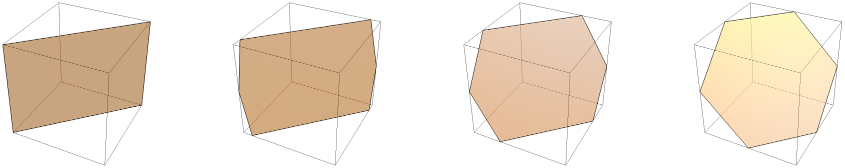

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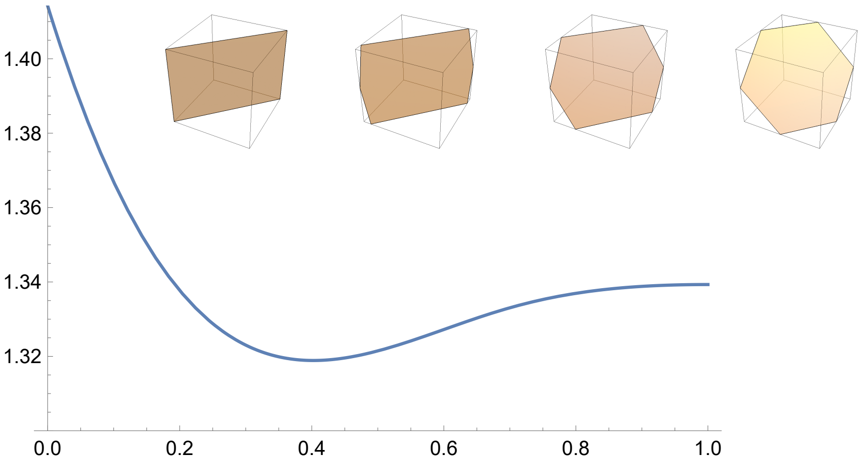

Non-diagonal critical directions

Consider the following class of vectors:

$$\mathrm{v}_{n,k}(a) := (\underbrace{a, \ldots\ldots a,}_{k} \; \underbrace{b, \ldots\ldots b}_{n-k}) \in S^{n-1},$$

where $a \in I_k := \left[0, \frac{1}{\sqrt{k}}\right]$, and $b := b_{n,k}(a) = \sqrt{\frac{1-ka^2}{n-k}}$.

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Main idea of the proof: there is some $a \in I_2$, for which the vector $\mathrm{v}_{n,2}(a)$ is non-diagonal, consists non-zero coordinates and is a critical direction.

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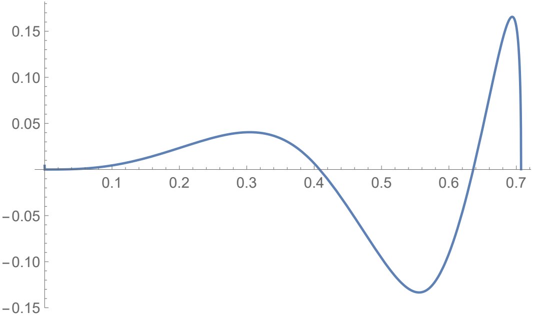
Based on the *Characterization Theorem*, $\mathrm{v}_{n,k}(a)$ is critical direction for exactly those a's, which are zeros of the function

$$F_{n,k}(a) := \frac{1}{1-a^2}\int_{-\infty}^{\infty} \operatorname{sinc}^{n-k} bt \cdot \operatorname{sinc}^{k-1} at \cdot \cos at \,\mathrm{d}t -$$

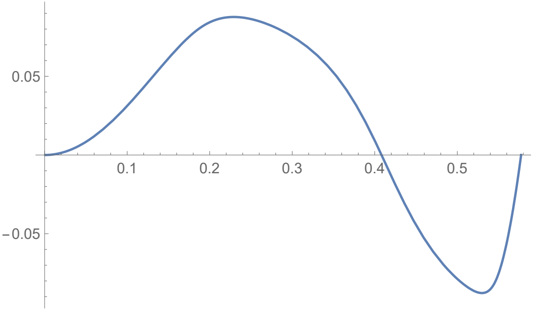
$$- \frac{1}{1-b^2}\int_{-\infty}^{\infty} \operatorname{sinc}^{n-k-1} bt \cdot \operatorname{sinc}^{k} at \cdot \cos bt \,\mathrm{d}t.$$

Building up the proof

graph of $F_{6,2}$



graph of $F_{6,3}$



Lemma 1

For each $2 \leq k \leq n-2$, $\dfrac{1}{\sqrt{k}}$ *and* $\dfrac{1}{\sqrt{n}}$ *are both zeros of* $F_{n,k}$.

Lemma 2

For each $4 \leq k \leq n-2$, $F_{n,k}$ *is differentiable* I_k. *In the case of* $k = 2, 3$, *differentiability is true for every compact subinterval of* I_k *which does not contain the right end point. Moreover, in both cases we have*

$$F'_{n,k}\left(\frac{1}{\sqrt{n}}\right) < 0.$$

Lemma 3

For each $n \geq 4$, $F_{n,2}(a) \geq 0$ *for every* $a \in \left[\gamma_n, \dfrac{1}{\sqrt{2}}\right]$ *where* $\gamma_n = \sqrt{\dfrac{n-2}{2n-3}}$.

Non-diagonal directions are not extremal

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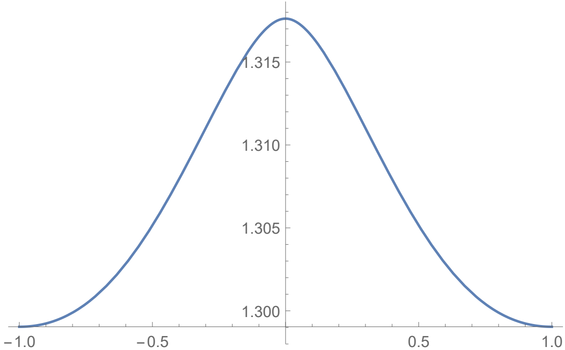
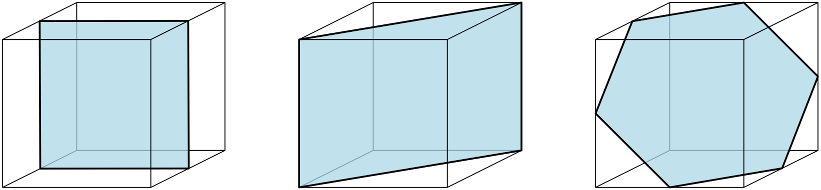
- Show that the Hessian matrix of the Lagrange function $\widetilde{H}$ is not positive definite at w.
- Find a vector q such that $\mathrm{q}\widetilde{H}\mathrm{q}^T < 0$.
- Suitable choice: $\mathrm{q} = (1, -1, 0, \ldots, 0)$.



Thank you for your attention!



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Open questions

- Number of zeros of $F_{n,2}$ and so the number of non-diagonal critical directions?
- Asymptotic behaviour of the zeros of $F_{n,2}$?
- Are there other type of critical directions?
- In the case of $k \geq 3$, are unit vectors $\mathrm{v}_{n,k}$ not critical directions?

The coordinates of critical directions are determined by the zeros of the function

$$F_{n,2}(a) = \frac{1}{1-a^2}\int_{-\infty}^{\infty} \operatorname{sinc}^{n-2} bt \cdot \operatorname{sinc} at \cdot \cos at \,\mathrm{d}t -$$

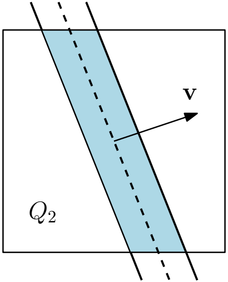
$$- \frac{1}{1-b^2}\int_{-\infty}^{\infty} \operatorname{sinc}^{n-3} bt \cdot \operatorname{sinc}^2 at \cdot \cos bt \,\mathrm{d}t.$$

In small dimensions these are:

4	0,632455	13	0,638774	22	0,639416
5	0,634265	14	0,638893	23	0,639453
6	0,636071	15	0,638998	24	0,639486
7	0,636935	16	0,639081	25	0,639517
8	0,637520	17	0,639156	26	0,639545
9	0,637921	18	0,639222	27	0,639570
10	0,638219	19	0,639278	28	0,639594
11	0,638445	20	0,639329	29	0,639616
12	0,638625	21	0,639375	30	0,639636

Tools from probability theory

Let $X_1, \dots, X_n$ be independent random variables distributed uniformly on $[-1, 1]$. The joint distribution $(X_1, \dots, X_n)$ induces the normalized Lebesgue measure on $2Q_n$. Let $\mathrm{v} \in S^{n-1}$.



$$\mathbb{P}\left(\left|\sum_{i=1}^{n} v_i X_i - r\right| \leq \varepsilon\right) = \frac{1}{2^n}\mathrm{Vol}_n(\mathrm{q} \in 2Q_n : \ |\langle \mathrm{q}, \mathrm{v}\rangle - r| \leq \varepsilon)$$

$$2f_{\sum_{i=1}^{n} v_i X_i}(r) = \frac{1}{|\mathrm{v}|} s\left(\mathrm{v}, \frac{r}{2}\right).$$

The characteristic function of the random variable $\sum_{i=1}^{n} v_i X_i$ is

$$\varphi_{\sum_{i=1}^{n} v_i X_i}(t) = \prod_{i=1}^{n} \operatorname{sinc}(v_i t),$$

where

$$\operatorname{sinc} x = \begin{cases} \frac{\sin x}{x}, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0. \end{cases}$$

Hence $s(\mathrm{v}, r)$ is derived by taking the inverse Fourier transform:

$$s(\mathrm{v}, r) = \frac{|\mathrm{v}|}{\pi} \int_{-\infty}^{\infty} \prod_{i=1}^{n} \operatorname{sinc}(v_i t) \cdot \cos(2rt) \, \mathrm{d}t.$$

Then the normalized central section $s(\mathrm{v}, 0)$ is

$$\sigma(\mathrm{v}) = \frac{|\mathrm{v}|}{\pi} \int_{-\infty}^{\infty} \prod_{i=1}^{n} \operatorname{sinc}(v_i t) \, \mathrm{d}t = s(\mathrm{v}, 0).$$

If $\mathsf{v} \in S^{n-1}$ is a critical direction, then

$$\frac{\partial}{\partial v_i}\sigma(\mathsf{v}) = -\sigma(\mathsf{v}) \cdot v_i.$$

Accordingly, v is a stationary point of the Lagrange function

$$\Lambda(\mathsf{v}) = \sigma(\mathsf{v}) + \tilde{\lambda}\big(|\mathsf{v}|^2 - 1\big)$$

where

$$\tilde{\lambda} = \frac{\sigma(\mathsf{v})}{2}$$

is the Lagrange multiplier.

Laplace-Pólya integral

Statement

For each $n \geq 2$

$$(n+3)J_{n+2}(0) < (n+2)J_n(0),$$

where

$$J_n(r) = \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{sinc}^n\, t \cdot \cos(rt)\,\mathrm{d}t$$

Connection of this integral with other fields:

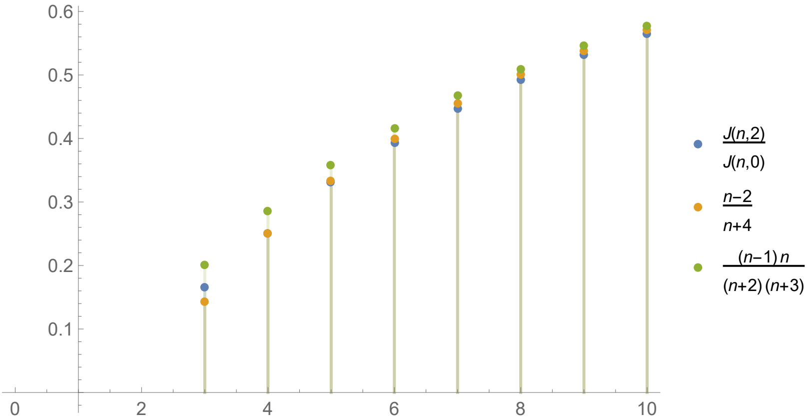
- geometry: volume of hyperplane sections of Q_n
- probability theory: probability density function of $\sum_{i=1}^{n} X_i$
- combinatorics: recursive formula by Thompson (1966)

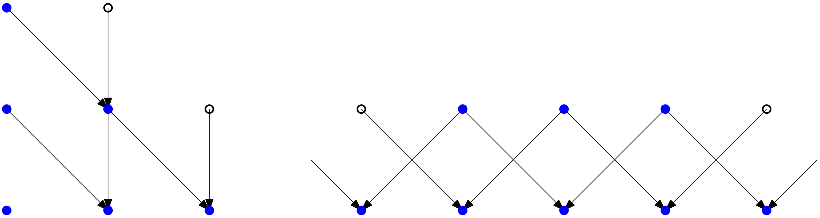
$$J_n(r) = \frac{n+r}{2(n-1)}J_{n-1}(r+1) + \frac{n-r}{2(n-1)}J_{n-1}(r-1)$$

Theorem

Let $n \geq 4$ and r be integers satisfying $-1 \leq r \leq n-2$. Then

$$\frac{J_n(r+2)}{J_n(r)} \leq \frac{(n-r-2)(n-r)(n-r+2)}{(n+r)(n+r+2)(n+r+4)}.$$





Recursive formula of Eulerian numbers:

$$A(m,l) = (m-l+1)A(m-1,l-1) + l\,A(m-1,l)$$

Recursive formula by Thompson (1966):

$$J_n(r) = \frac{n+r}{2(n-1)} J_{n-1}(r-1) + \frac{n-r}{2(n-1)} J_{n-1}(r-1)$$

Connection between them:

$$J_n(r) = \frac{1}{(n-1)!} A\left(n-1, \frac{n+r}{2}\right)$$