Banach-embeddable diversities

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*PID2022-136320NB-I00 project / AEI/10.13039/501100011033/ FEDER, UE

Universidad de Murcia

Discrete Geometry Days^3
3rd July 2024

Diversity

X, $\delta : \mathcal{P}_F(X) \to [0, +\infty)$ *diversity* if

- $\delta(A) = 0$ iff $|A| \leq 1$
- $\delta(A \cup C) \leq \delta(A \cup B) + \delta(B \cup C)\ \forall\ |B| \geq 1.$

Metric spaces

- (X, δ) diversity $\Rightarrow d(x, y) := \delta(\{x, y\})$ distance

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- (X, d) metric $\Rightarrow \delta(A) := \sum_{x,y \in A} d(x,y)$ diversity
  $\delta(A) := \max_{x,y \in A} d(x,y)$ diversity

Circumradius

$K, C \in \mathcal{K}^n$ then

$$R(K, C) = \inf\{\rho \geq 0 : x + K \subset \rho C,\ x \in \mathbb{R}^n\}$$

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Diversities and circumradius

$X \subset \mathcal{P}_F(\mathbb{R}^n)$, $C \in \mathcal{K}^n \ \Rightarrow \ \delta(X) := R(X, C)$ diversity

Minkowski diversity

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- δ sublinear: $\delta(X+Y) \leq \delta(X) + \delta(Y)$
  $\delta(\lambda X) = \lambda \delta(X) \;\; \forall \lambda > 0$

Minkowski diversities

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- δ sublinear: $\delta(X+Y) \leq \delta(X) + \delta(Y)$
  $\delta(\lambda X) = \lambda\delta(X) \;\; \forall \lambda > 0$
- $\forall$ A, B, $\exists$ a, b s.t.

$$\delta((a+A) \cup (b+B)) \leq \max\{\delta(A), \delta(B)\}$$

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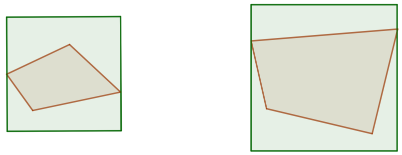

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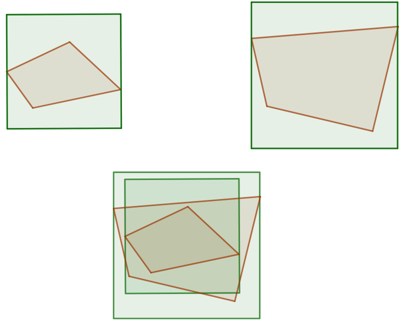

Theorem (D.Bryant, K.T.Huber, V.Moulton, P.F.Tupper '21)

$(\mathbb{R}^n, \delta)$ diversity. Equivalent:

1. δ is Minkowski diversity
2. - δ sublinear: $\delta(X+Y) \leq \delta(X) + \delta(Y)$
     $\delta(\lambda X) = \lambda\delta(X) \quad \forall\, \lambda > 0$
   - $\forall$ A, B, $\exists$ a, b s.t.

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Banach diversities (G.M. '22)

$X \in \mathcal{P}_F(\mathbb{R}^n)$, $C \in \mathcal{K}_0^n$

$$\delta(X) := R(X, C)$$

is a *Banach diversity*

Theorem (G.M. '22)

$(\mathbb{R}^n, \delta)$ diversity. Equivalent:

1. δ is Banach diversity
2. - δ seminorm: $\delta(X+Y) \leq \delta(X) + \delta(Y)$
   $\delta(\lambda X) = |\lambda|\delta(X) \quad \forall \lambda \in \mathbb{R}$
   - $\forall$ A, B, $\exists$ a, b s.t.

$$\delta((a+A) \cup (b+B)) \leq \max\{\delta(A), \delta(B)\}$$

Minkowski (Banach) embeddable

(X, δ) diversity, $|X| = m \in \mathbb{N}$ if

$$\exists C \in \mathcal{K}^n \ (\in \mathcal{K}^n_0) \text{ and } p_i \in \mathbb{R}^n,\ i = 1, \ldots, m$$

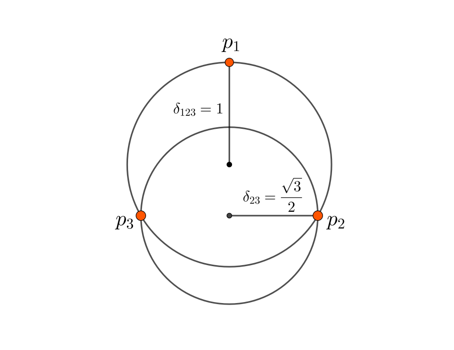
s.t.

$$\delta_{i_1,\ldots,i_k} := \delta(x_{i_1}, \ldots, x_{i_k}) = R(\{p_{i_1}, \ldots, p_{i_k}\}, C)$$

δ is Minkowski (Banach) embeddable

Example

$\delta_{12} = \delta_{13} = \delta_{23} = \frac{\sqrt{3}}{2}$, $\delta_{123} = 1$



$C = \mathbb{B}_2$ and p_1, p_2, p_3 equilateral triangle

Propierties

(X, δ) diversity, $|X| = 3$

$$0 = \delta_I < \delta_{ij} \leq \delta_{123} \leq \delta_{ij} + \delta_{jk}$$

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Theorem (R.Brandenberg, S.König '13)

$$R(\{p_1, p_2, p_3\}, C) \leq R(\{p_1, p_2\}, C) + R(\{p_2, p_3\}, C)$$

Theorem (D.Bryant, K.T.Huber, V.Moulton, P.F.Tupper '21)

(X, δ) diversity, $|X| = 3 \;\Rightarrow\;$ δ Minkowski embeddable

Theorem (G.M. '22)

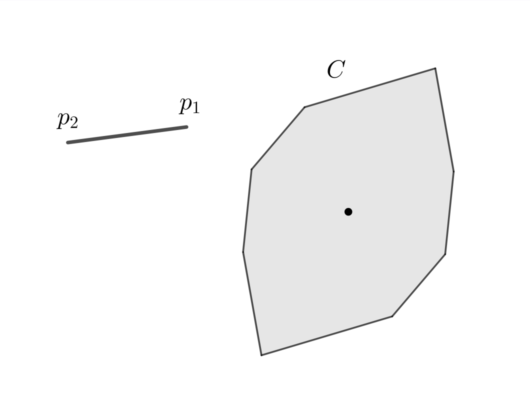
(X, δ) diversity, $|X| = 3$ with $0 < \delta_{13} \leq \delta_{12}$. Equivalent:

1. δ is Banach diversity
2. $$\delta_{12} - \delta_{13} \leq \delta_{23} \leq \delta_{12} + \delta_{13}$$
   $$\delta_{ij} \leq \delta_{123} \leq \frac{8\delta_{12}\delta_{13}\delta_{23}}{\sqrt{3}(2(\delta_{12}\delta_{13} + \delta_{12}\delta_{23} + \delta_{13}\delta_{23}) - \delta_{12}^2 - \delta_{13}^2 - \delta_{23}^2)}$$

Lemma

Let $C \in \mathcal{K}_0^n$, $p_1, p_2 \in \mathbb{R}^n$. Then

$$\frac{1}{2R([p_1, p_2], C)}[p_1 - p_2, p_2 - p_1] \subset C$$



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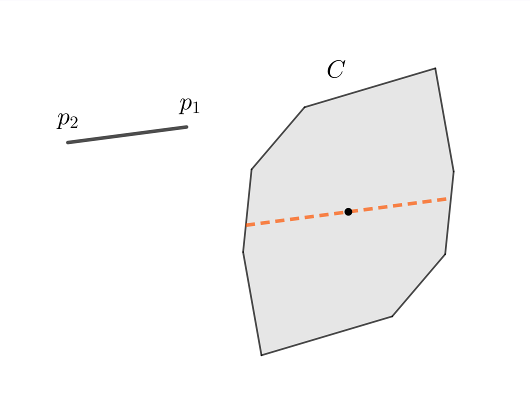

Proof of Banach embeddability for $|X| = 3$

p_1, p_2, p_3 equilateral triangle

$$p_1 = \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right), \quad p_2 = \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right), \quad \text{and} \quad p_3 = (0, 1)$$

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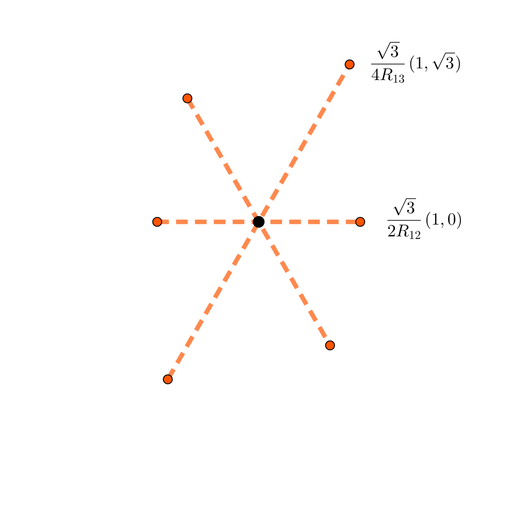
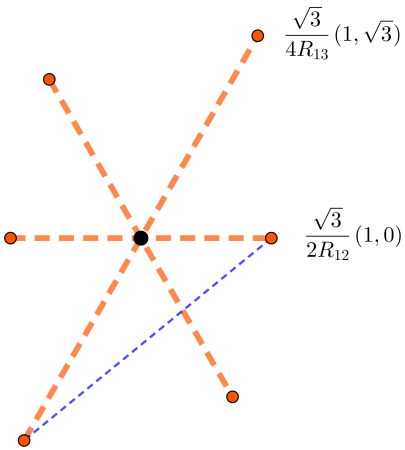
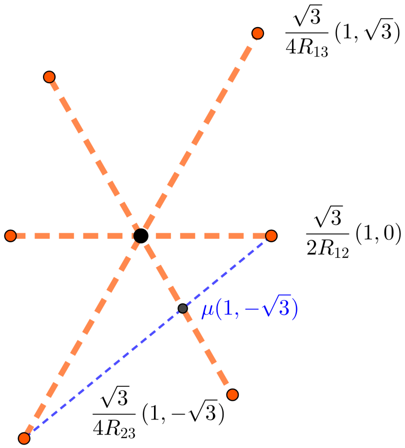
Reordering to $0 < R_{13} \leq R_{12}$

Using the Lemma:

$$\pm\frac{\sqrt{3}}{2R_{12}}(1,0), \pm\frac{\sqrt{3}}{4R_{13}}(1,\sqrt{3}), \pm\frac{\sqrt{3}}{4R_{23}}(1,-\sqrt{3}) \in \partial C$$




$\frac{\sqrt{3}}{4R_{13}}(1,\sqrt{3})$
$\frac{\sqrt{3}}{2R_{12}}(1,0)$

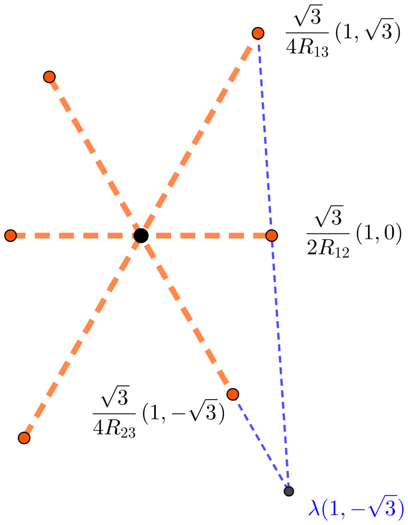


$\frac{\sqrt{3}}{4R_{13}}(1,\sqrt{3})$
$\frac{\sqrt{3}}{2R_{12}}(1,0)$
$\mu(1,-\sqrt{3})$
$\frac{\sqrt{3}}{4R_{23}}(1,-\sqrt{3})$

$$\mu = \frac{\sqrt{3}}{4(R_{12} + R_{13})}$$

and thus

$$\frac{\sqrt{3}}{4(R_{12} + R_{13})} \leq \frac{\sqrt{3}}{4R_{23}}$$

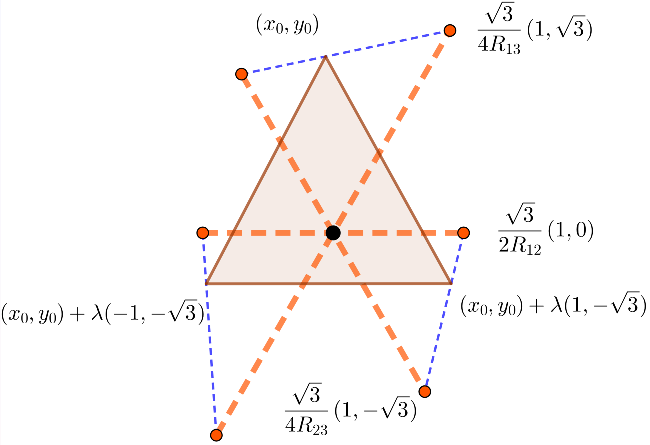


$\frac{\sqrt{3}}{4R_{13}}(1,\sqrt{3})$
$\frac{\sqrt{3}}{2R_{12}}(1,0)$
$\frac{\sqrt{3}}{4R_{23}}(1,-\sqrt{3})$
$\lambda(1,-\sqrt{3})$

$$\lambda = \frac{\sqrt{3}}{4(R_{12} - R_{13})}$$

and thus

$$\frac{\sqrt{3}}{4R_{23}} \leq \frac{\sqrt{3}}{4(R_{12} - R_{13})}$$



(x_0, y_0)
$\frac{\sqrt{3}}{4R_{13}}(1, \sqrt{3})$
$\frac{\sqrt{3}}{2R_{12}}(1, 0)$
$(x_0, y_0) + \lambda(-1, -\sqrt{3})$
$(x_0, y_0) + \lambda(1, -\sqrt{3})$
$\frac{\sqrt{3}}{4R_{23}}(1, -\sqrt{3})$

Letting

$$a = \frac{\sqrt{3}}{2R_{12}}, \quad b = \frac{\sqrt{3}}{2R_{13}}, \quad c = \frac{\sqrt{3}}{2R_{23}},$$

then

$$(x_0, y_0) = (1 - t_1)b\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) + t_1 c\left(\frac{-1}{2}, \frac{-\sqrt{3}}{2}\right),$$

$$(x_0, y_0) + \lambda\left(1, -\sqrt{3}\right) = (1 - t_2)a(1, 0) + t_2 c\left(\frac{1}{2}, \frac{-\sqrt{3}}{2}\right),$$

$$(x_0, y_0) + \lambda\left(-1, -\sqrt{3}\right) = (1 - t_3)a(-1, 0) + t_3 b\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right),$$

$$\lambda = \frac{2abc(a+b) - a^2b^2 - c^2(a-b)^2}{4abc},$$

$$x_0 = \frac{(a-b)c^2 - b^2(c-a)}{4bc},$$

$$y_0 = \frac{(2\sqrt{3}ab + \sqrt{3}b^2)c - \sqrt{3}ab^2 - (\sqrt{3}a - \sqrt{3}b)c^2}{4bc},$$

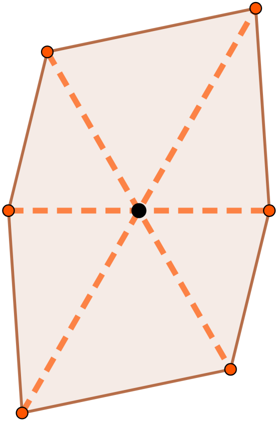
$$t_1 = \frac{ab - (a-b)c}{2bc},$$

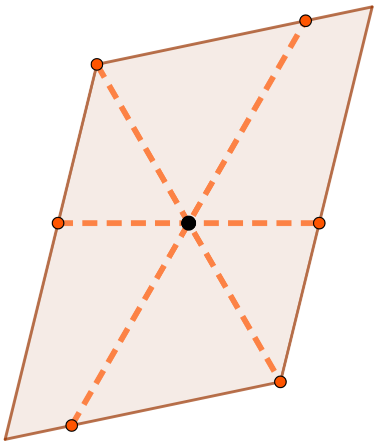
$$t_2 = \frac{ab + (a-b)c}{2ac},$$

$$t_3 = \frac{ab + (a-b)c}{2ab}.$$

As a consequence

$$R_{123} \leq \frac{1}{\lambda} = \frac{4abc}{2abc(a+b) - a^2b^2 - c^2(a-b)^2}$$





$$
\begin{aligned}
\lambda &= \frac{2abc(a+b) - a^2b^2 - c^2(a-b)^2}{4abc} \geq 0,\\
x_0 &= \frac{(a-b)c^2 - b^2(c-a)}{4bc},\\
y_0 &= \frac{(2\sqrt{3}ab + \sqrt{3}b^2)c - \sqrt{3}ab^2 - (\sqrt{3}a - \sqrt{3}b)c^2}{4bc},\\
t_1 &= \frac{ab - (a-b)c}{2bc} \in [0,1],\\
t_2 &= \frac{ab + (a-b)c}{2ac} \in [0,1],\\
t_3 &= \frac{ab + (a-b)c}{2ab} \in [0,1].
\end{aligned}
$$

Lemma

$$f(x,y,z) := 2xy + 2xz + 2yz - x^2 - y^2 - z^2 \geq 0$$

under conditions

$$0 \leq y \leq x, \quad x - y \leq z \leq x + y$$

$$(x = R_{12}, \quad y = R_{13}, \quad z = R_{23})$$

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Proof

Schmüdgen's Positivstellensatz

$$\begin{aligned} &2xy + 2xz + 2yz - x^2 - y^2 - z^2 \\ &= (z - x + y)(x + y - z) + 4y(x - y) + 2y(z - x + y) \end{aligned}$$

Concluding remarks

- δ s.t. $\delta_{12} = \delta_{13} = 2$, $\delta_{23} = 1$,

$$\delta_{123} \leq \min\{3, 3, 4\} \;\; \not\equiv \;\; \delta_{123} \leq \frac{4}{\sqrt{3}}$$

Concluding remarks

- δ s.t. $\delta_{12} = \delta_{13} = 2$, $\delta_{23} = 1$,

$$\delta_{123} \leq \min\{3, 3, 4\} \quad \not\equiv \quad \delta_{123} \leq \frac{4}{\sqrt{3}}$$

- Characterizing *Minkowski embeddability* $|X| \geq 4$?

Banach embeddability

$$R_{ijk} \leq R_{1234} \leq 2\left[R_{13}^2 R_{14} R_{24}^2 + R_{12}^2 R_{14} R_{34}^2 + (R_{12}R_{13}R_{14} - (R_{12}+R_{13})R_{14}^2)R_{23}^2 - (R_{12}R_{13}^2 + R_{13}R_{14}^2 - (R_{12}R_{13} + R_{13}^2)R_{14})R_{23}R_{24} - (2R_{12}R_{13}R_{14}R_{24} - (R_{12}^2 R_{13} + R_{12}R_{14}^2 - (R_{12}^2 + R_{12}R_{13})R_{14})R_{23})R_{34}\right] /$$

$$\left[2R_{13}R_{14}R_{24}^2 + (R_{12}R_{13} - (R_{12}+R_{13})R_{14} - R_{14}^2)R_{23}^2 + (R_{12}^2 R_{13} - R_{12}R_{13}^2 - (R_{12}+R_{13})R_{14}^2 - (R_{12}^2 - 2R_{12}R_{13} - R_{13}^2)R_{14})R_{23} - (R_{12}^2 R_{13} + R_{12}R_{13}^2 - (R_{12}-R_{13})R_{14}^2 - (R_{12}^2 + R_{13}^2)R_{14} + (R_{12}R_{13} - (R_{12}+R_{13})R_{14} + R_{14}^2)R_{23})R_{24} + (R_{12}^2 R_{13} + R_{12}R_{13}^2 + (R_{12}-R_{13})R_{14}^2 - 2R_{12}R_{14}R_{24} + (R_{12}^2 - 2R_{12}R_{13} - R_{13}^2)R_{14} - (R_{12}R_{13} - (R_{12}+R_{13})R_{14} - R_{14}^2)R_{23})R_{34}\right]$$

- R. Brandenberg, S. König, No Dimension-Independent Core-Sets for Containment Under Homothetics, Discr. Comput. Geom. 49 (2013), no. 1, 3–21.
- D. Bryant, K. T. Huber, V. Moulton, P. F. Tupper, Diversities and the Generalized Circumradius, Discr. Comput. Geom. 70 (2023), no. 4, 1862–1883.
- B. González Merino, On diversities and finite dimensional Banach spaces, J. Conv. Anal. 2024.

Thank you for your attention!!