Improving Inequalities on k-Dimensional Volume Extremal Ellipsoids Using Asymmetry Coefficients

Joint work with René Brandenberg

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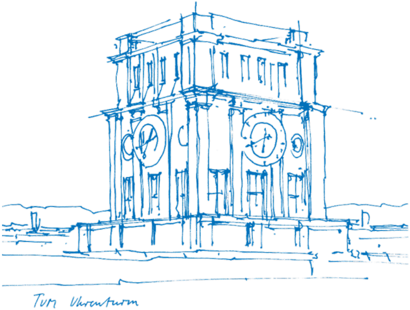

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Any $C \in \mathcal{K}^n$ contains a unique volume-maximal ellipsoid $\mathcal{E}_J(C)$ (resp. is contained in a unique volume-minimal ellipsoid $\mathcal{E}_L(C)$).

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    - (a) $\sum_{i=1}^{m} \lambda_i u_i = 0$, and

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- (a) $\sum_{i=1}^{m} \lambda_i u_i = 0$, and
- (b) $\sum_{i=1}^{m} \lambda_i (u_i u_i^T) = I_n$.

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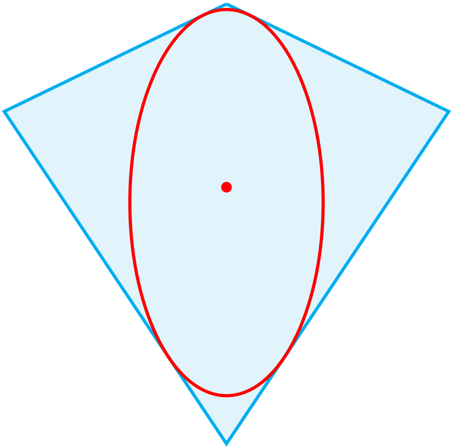
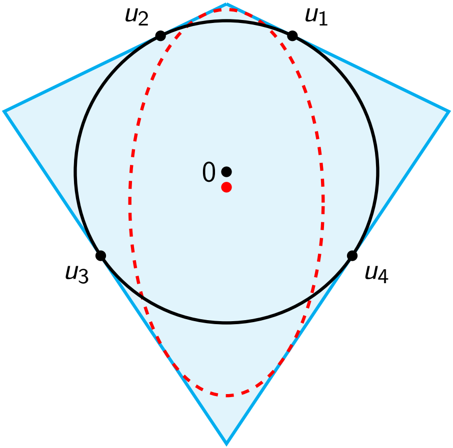
(b) implies $\sum_{i=1}^{m} \lambda_i = n$.





u_2
u_1
0
u_3
u_4

Corollary (John, 1948)

For $C \in \mathcal{K}^n$ and c_J the center of $\mathcal{E}_J(C)$:

$$(\mathcal{E}_J(C) - c_J) \subset (C - c_J) \subset n \cdot (\mathcal{E}_J(C) - c_J).$$

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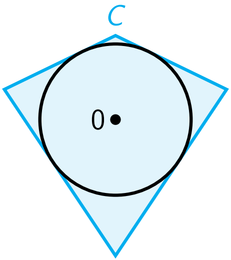
$$d_{BM}(C, \mathbb{B}_2^n) \leq \begin{cases} \sqrt{n} & \text{, if } C \text{ is symmetric} \\ n & \text{, else.} \end{cases}$$

Definition

The John asymmetry of $C \in \mathcal{K}^n$ is

$$s_J(C) := \min\{\rho \geq 0 : (C - c_J) \subset \rho(c_J - C)\},$$

where c_J is the center of $\mathcal{E}_J(C)$.



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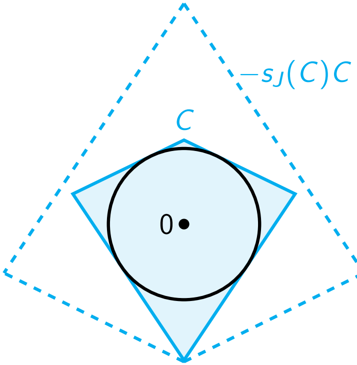
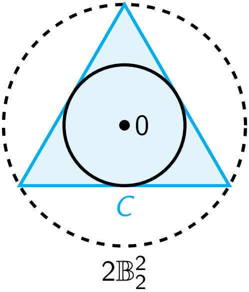


- $1 \leq s_J(C) \leq n$ by John's theorem,



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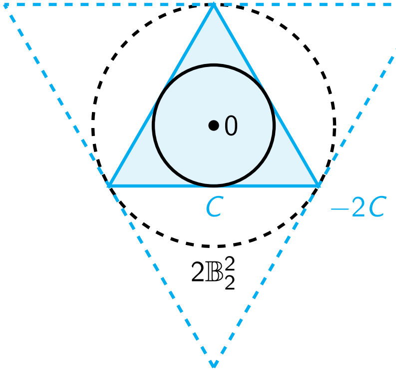
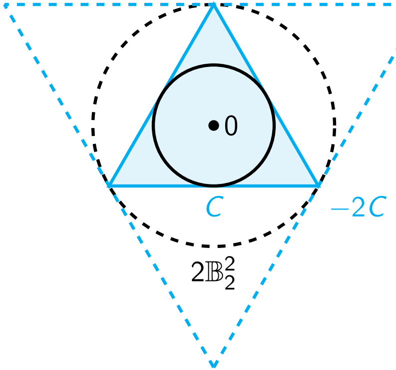


- $1 \leq s_J(C) \leq n$ by John's theorem,
- $s_J(C) = 1 \iff C$ is symmetric,



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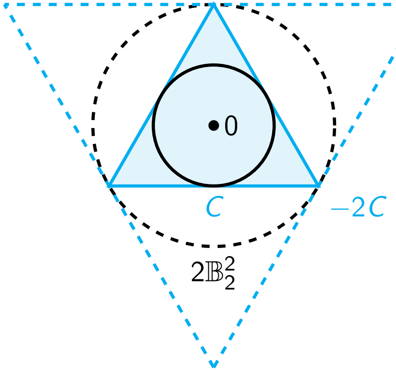

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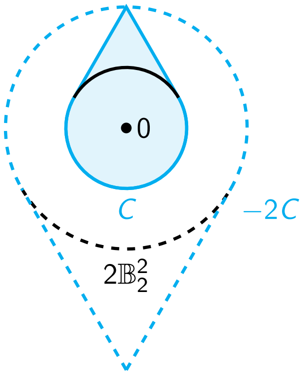


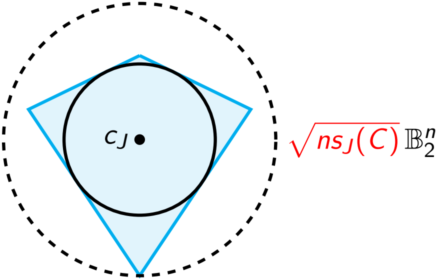
Theorem (Brandenberg, König, 2013)

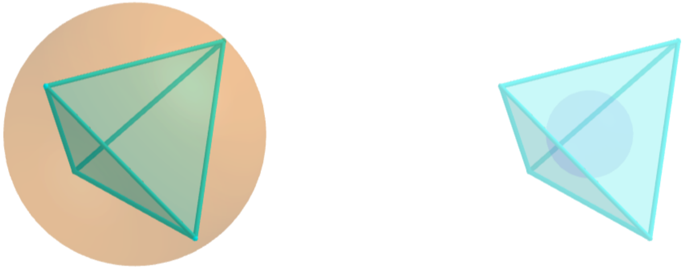
For $C \in \mathcal{K}^n$ and c_J the center of $\mathcal{E}_J(C)$:

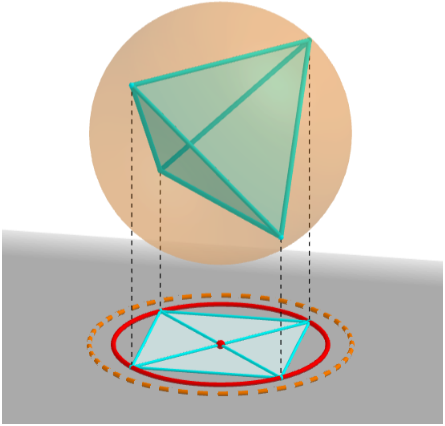
$$(\mathcal{E}_J(C) - c_J) \subset (C - c_J) \subset \sqrt{ns_J(C)}\,(\mathcal{E}_J(C) - c_J).$$

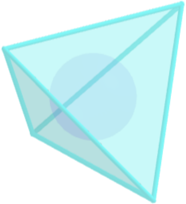
This bound is best possible for all values $s_J(C) \in [1, n]$.

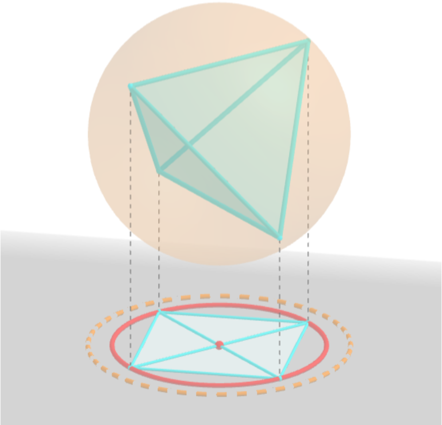
In particular $d_{BM}(K, \mathbb{B}_2^n) \leq \sqrt{ns_J(C)}$.

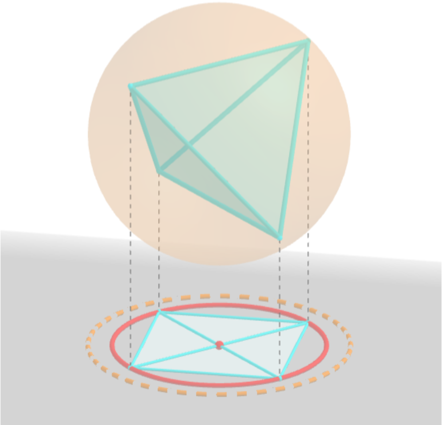


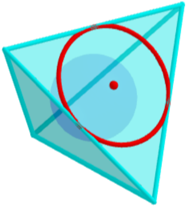


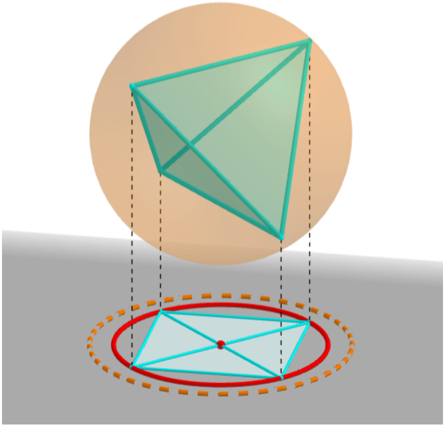


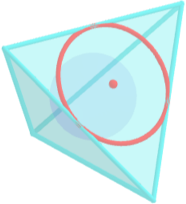












$C|F$ orthogonal projection of $C \in \mathcal{K}^n$ onto $F \subset \mathbb{R}^n$ linear k-space

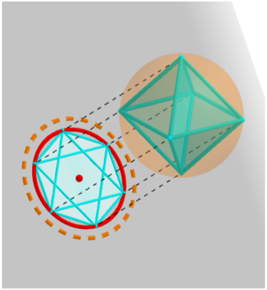
Theorem (BG, 2024+)

For $C \in \mathcal{K}^n$ with $\mathcal{E}_L(C) = \mathbb{B}_2^n$ and $F \subset \mathbb{R}^n$ linear k-space:

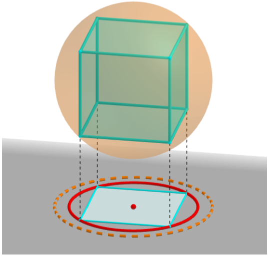
$$\mathrm{vol}_k(\mathcal{E}_L(C|F)) \geq \mathrm{vol}_k\left(\sqrt{\frac{k}{n}}\mathbb{B}_2^k\right).$$

Equality holds if and only if $\mathcal{E}_L(C|F) = \sqrt{\frac{k}{n}}\mathbb{B}_2^n|F$.

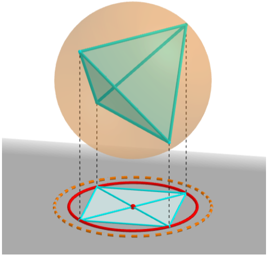
Equality holds for appropriate k-spaces e.g. if C is



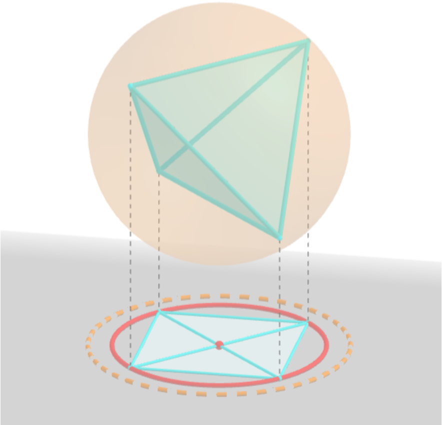
a cross-polytope

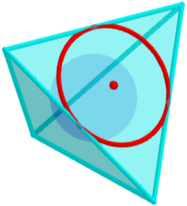


a cube



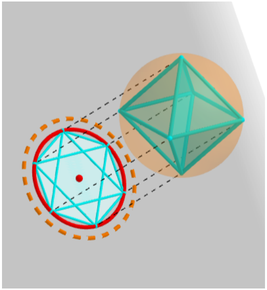
an n-simplex (unless n even and $k \in \{1, n-1\}$)

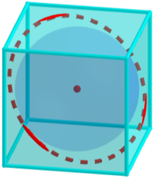




Polar of $C \in \mathcal{K}^n$ with $0 \in \mathrm{int}(C)$:

$$C^\circ = \{a \in \mathbb{R}^n : a^T x \leq 1 \text{ for all } x \in C\} \in \mathcal{K}^n.$$





$\mathcal{E}_L(C^\circ) = \mathbb{B}_2^n$ $\Longleftrightarrow$ $\mathcal{E}_J(C) = \mathbb{B}_2^n$

$E^\circ \supset C^\circ$ cylinder with k-ellipsoidal base $\Longleftrightarrow$ $E \subset C$ k-ellipsoid with $0 \in \mathrm{relint}(E)$

Corollary

For $C \in \mathcal{K}^n$ symmetric with $\mathcal{E}_J(C) = \mathbb{B}_2^n$ and $E \subset C$ a k-ellipsoid:

$$vol_k(E) \leq vol_k\left(\sqrt{\frac{n}{k}}\mathbb{B}_2^k\right).$$

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Theorem (Ball, 1992)

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$$vol_k(E) \leq vol_k\left(\sqrt{\frac{n(n+1)}{k(k+1)}}\mathbb{B}_2^k\right).$$

Equality holds if E is the inscribed k-ball of a k-face of a regular simplex.

Theorem (BG, 2024+)

For $C \in \mathcal{K}^n$ with $\mathcal{E}_J(C) = \mathbb{B}_2^n$ and $E \subset C$ a k-ellipsoid:

$$vol_k(E) \leq vol_k\left(\sqrt{\frac{n}{k} \cdot \underbrace{\min\left\{\frac{n+1}{k+1}, \frac{s_J(C)+1}{2}\right\}}_{=:\,\mathrm{m}(s_J(C))}}\,\mathbb{B}_2^k\right).$$

Equality holds if and only if E is a k-ball of radius $\sqrt{\frac{n}{k} \cdot \mathrm{m}(s_J(C))}$.

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Equality holds if and only if E is a k-ball of radius $\sqrt{\frac{n}{k} \cdot \mathrm{m}(s_J(C))}$.

In the equality case:

- The center c of E is perp. to $\mathrm{aff}(E)$ with $\|c\| = \sqrt{n \cdot (\mathrm{m}(s_J(C)) - 1)}$.

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- For $u_1, ..., u_m$ from John's theorem: $c^T u_i \in \{1, 1 - \mathrm{m}(s_J(C))\}$.

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Equality holds if and only if E is a k-ball of radius $\sqrt{\frac{n}{k} \cdot \mathrm{m}(s_J(C))}$.

This inequality is best possible if and only if $s_J(C) \notin (1, 1 + \frac{2}{n})$.

In the equality case:

- The center c of E is perp. to aff(E) with $\|c\| = \sqrt{n \cdot (\mathrm{m}(s_J(C)) - 1)}$.
- For $u_1, ..., u_m$ from John's theorem: $c^T u_i \in \{1, 1 - \mathrm{m}(s_J(C))\}$.

$$vol_k(E) \leq vol_k\left(\sqrt{\frac{n}{k}\cdot\frac{n+1}{k+1}}\mathbb{B}_2^k\right)$$

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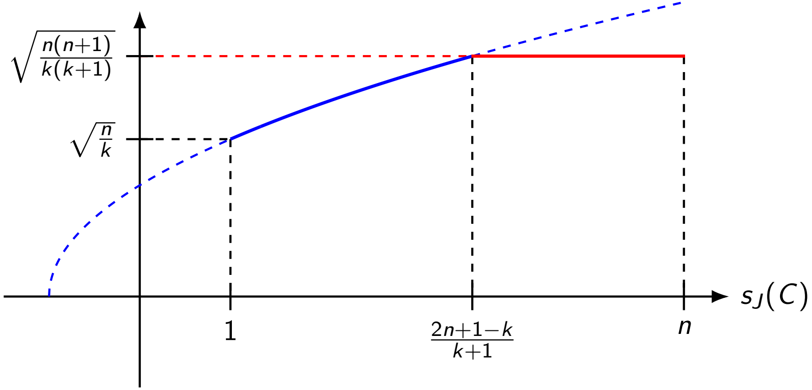

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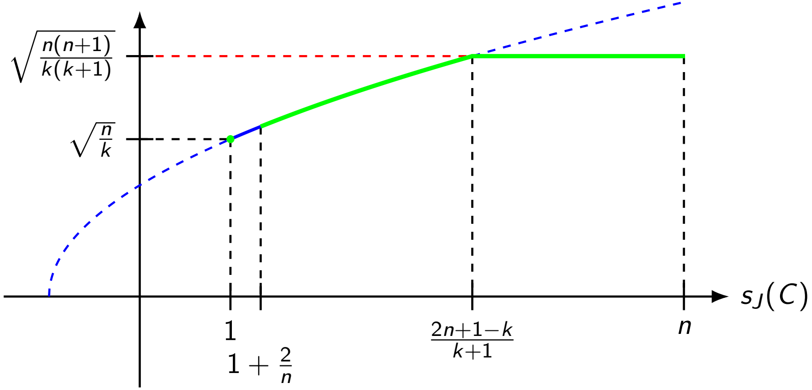
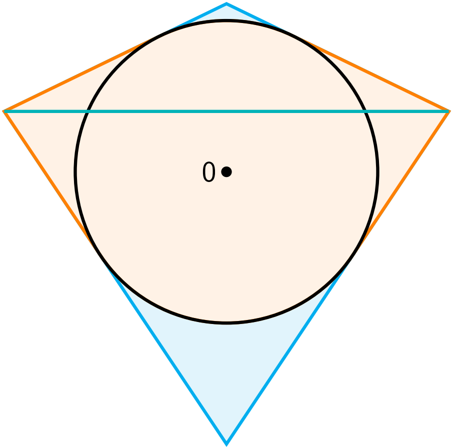


Best we know for $s_J \in \left(1, 1 + \frac{2}{n}\right)$:

There exists $C \in \mathcal{K}^n$ with $\mathcal{E}_J(C) = \mathbb{B}^n$ and $s_J(C) = s_J$:

$$\sqrt{\frac{n}{k} \cdot \frac{4(k(s_J+1)+1)+n(s_J-1)^2+n\sqrt{((s_J-1)(s_J+3)-\frac{4}{n})^2+8(1+\frac{2}{n}-s_J)(s_J^2-1)\frac{n-k}{n}}}{8(k+1)}}\,\mathbb{B}_2^k \subseteq_t C$$



$$vol_k(E) \leq vol_k\left(\sqrt{\frac{n}{k} \cdot \frac{n+1}{k+1}}\mathbb{B}_2^k\right)$$

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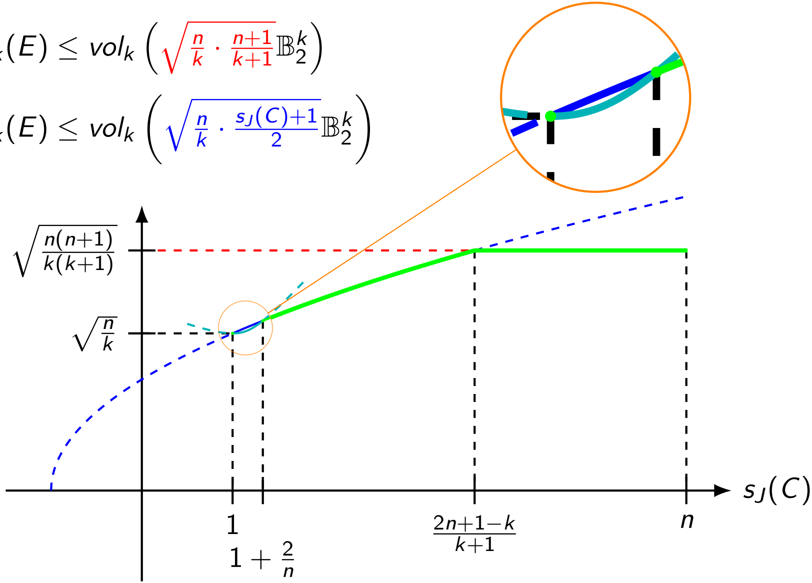

Theorem (BG, 2024+)

For $C \in \mathcal{K}^2$ with $\mathcal{E}_J(C) = \mathbb{B}_2^2$ and $x, y \in C$:

$$\underbrace{\|x-y\|}_{=\,\mathrm{vol}_1([x,y])} \leq \underbrace{\sqrt{s_J(C)^2 + 5 + \sqrt{4(2-s_J(C))^2 + (s_J(C)^2-1)^2}}}_{=:\,\mathrm{d}(s_J(C))}.$$

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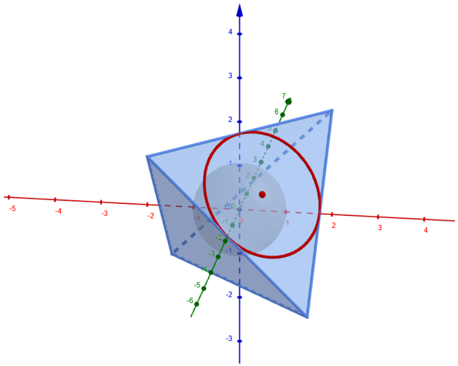
Thank you for your attention!

For $s_J \in \left[\frac{2n+1-k}{k+1}, n\right]$:

$T \in \mathcal{K}^n$ a centered simplex, $C := T \cap (-s_J \cdot T)$,

E inscribed k-ball of a facet of T



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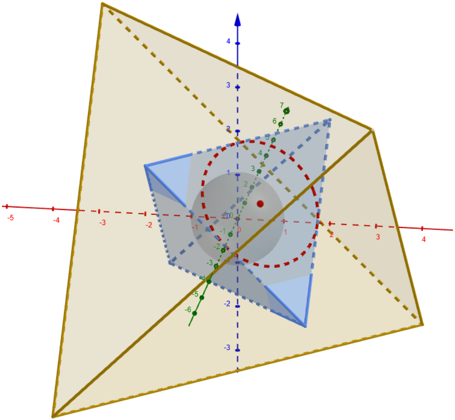

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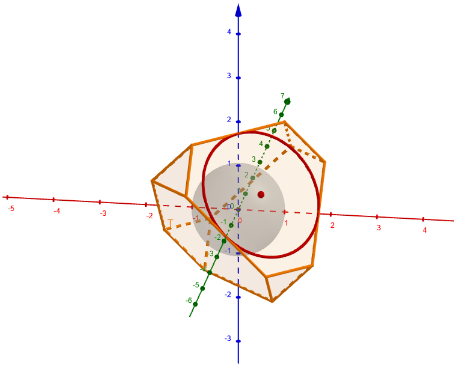
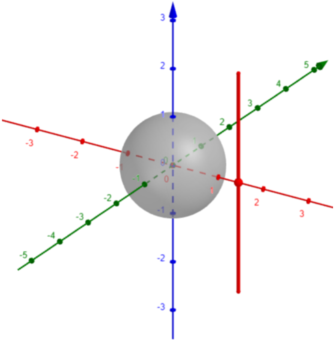


For $s_J \in \left[1 + \frac{2}{n}, \frac{2n+1-k}{k+1}\right]$:

(i) E is a k-ball of radius $\sqrt{\frac{n}{k} \cdot \frac{s_J+1}{2}}$.

(ii) The center c of E is perp. to aff(E) with $||c|| = \sqrt{n \cdot \frac{s_J-1}{2}}$.

(iii) For $u^1, \dots, u^m$ from John's theorem: $c^T u^i \in \left\{1, \frac{1-s_J}{2}\right\}$.



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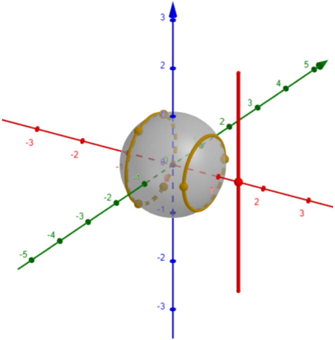

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