# On the number of points a given circle can cover from a diameter one finite point set

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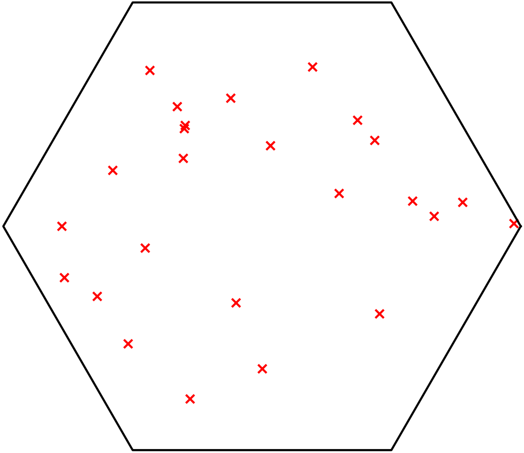

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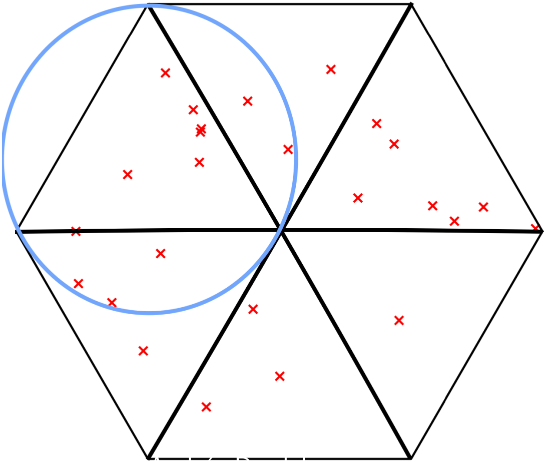

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Let $N$ be the largest integer such that in any **diameter** 1 **set** of $3n$ **points** we can cover **at least** $N$ of them with a circle with radius $r$ . Prove that there exists an $\epsilon > 0$ (depending on $n$) such that the value of $N$ does not depend on $r$ in the interval $r \in (\frac{1}{2} - \epsilon, \frac{1}{2})$.

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**Problem (Generalization of the hexagonal dart board problem)**

*Let $n$ be a fixed positive integer. Let $\mathcal{P}_n$ be the family of all sets of $n$ points, so that in any set, the distance between any two points is at most 1. Let the function value $N_n(r)$ $(0 < r \leq 1)$ be the largest integer $k$ so that for every point set $P \in \mathcal{P}_n$ there exists a circle of radius $r$ which covers at least $k$ points in $P$.*

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We especially want to know what happens as $n$ gets large. Let

$$c(r) := \lim_{n \to \infty} \frac{N_n(r)}{n}$$

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- $N_{3n}(r) = n$ on the interval $r \in (\frac{1}{2} - \epsilon, \frac{1}{2})$.
- This implies that $c(r) = \frac{1}{3}$ on this interval. We will see that this is true on a much larger interval!

# Jung's Theorem

**Theorem (Jung's Theorem (planar version))**

*Every diameter $d$ point set can be covered by a circle of radius $r \leq \frac{d}{\sqrt{3}}$.*

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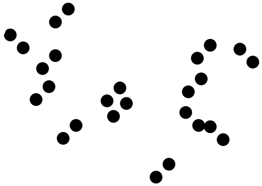

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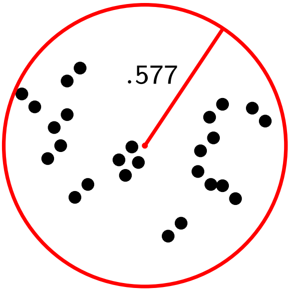

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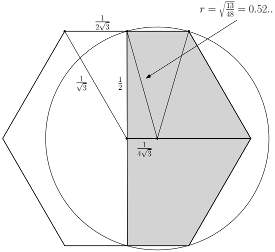

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"...it does seem safe to guess that progress on [this problem], which has been painfully slow in the past, may be even more painfully slow in the future." Klee and Wagon

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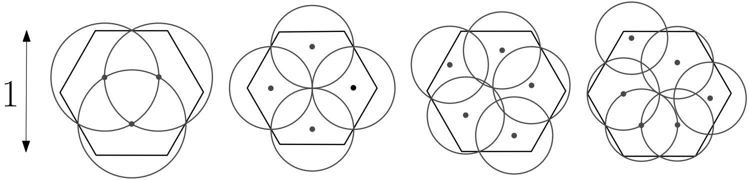
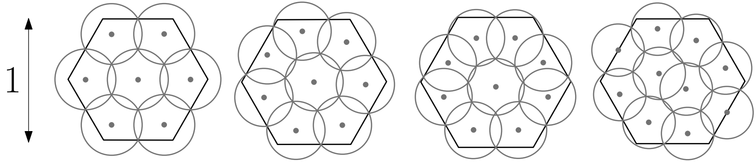



$k = 3$ — $N_n(0.433) \geq \lceil \frac{n}{3} \rceil$

$k = 4$ — $N_n(0.366) \geq \lceil \frac{n}{4} \rceil$

$k = 5$ — $N_n(0.321) \geq \lceil \frac{n}{5} \rceil$

$k = 6$ — $N_n(0.288) \geq \lceil \frac{n}{6} \rceil$




$k = 7$ — $N_n(0.25) \geq \lceil \frac{n}{7} \rceil$

$k = 8$ — $N_n(0.230) \geq \lceil \frac{n}{8} \rceil$

$k = 9$ — $N_n(0.217) \geq \lceil \frac{n}{9} \rceil$

$k = 10$ — $N_n(0.209) \geq \lceil \frac{n}{10} \rceil$

Work of: Y. Liu 2022

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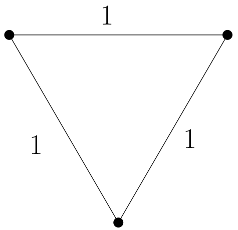


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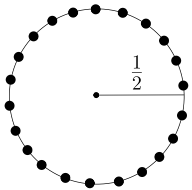

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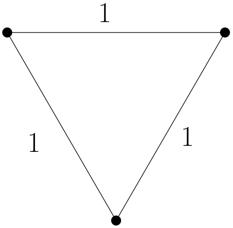


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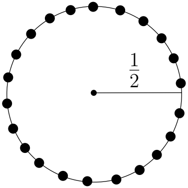


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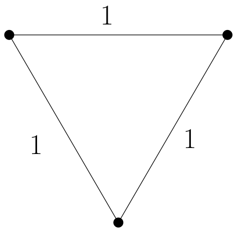


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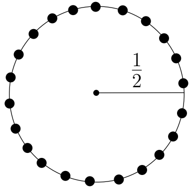


Implications:

- Ex 1: $N_n(r) \leq \lceil \frac{n}{3} \rceil$ on $r \in (0, \frac{1}{2})$ and $N_n(r) \leq \lceil \frac{2n}{3} \rceil$ on $r \in [\frac{1}{2}, \frac{1}{\sqrt{3}})$

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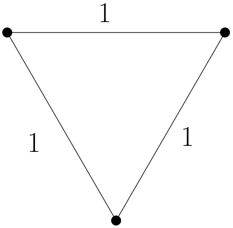


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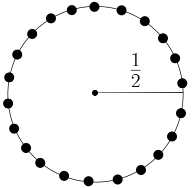


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- Ex 2: $N_n(r) \leq \lceil \frac{n}{\pi} \sin^{-1}(2r) \rceil$ for $r \in (0, \frac{1}{2})$

# Current knowledge of $N_n(r)$ and $c(r)$

| | Labels | Interval | Decimal Approx. | $N_n(r)$ result |
|---|---|---|---|---|
| Theorem 1 | Interval **A** | $r \in [\frac{\sqrt{3}}{3}, 1]$ | $r \in [.577, 1]$ | $N_n(r) = n$ |
| Theorem 2 | Interval **B** | $r \in [\frac{\sqrt{3}}{4}, \frac{1}{2})$ | $r \in [.433, .5]$ | $N_n(r) = \lceil \frac{n}{3} \rceil$ |
| Theorem 3 | Point **b** | $r = \frac{1}{2}$ | $r = .5$ | $n \leq N_{3n}(r) \leq n + 1$ |
| Theorem 4 | Point **c** | $r = \frac{\sqrt{2}}{4}$ | $r \approx .354$ | $N_n(r) = \lceil \frac{n}{4} \rceil$ |
| Theorem 5 | Point **d** | $r = \frac{1}{4}$ | $r = .25$ | $N_n(r) = \lceil \frac{n}{7} \rceil, n \neq 7$ |
| Example 1 | Interval **E** | $r \in [\frac{\sqrt{3}-1}{\sqrt{2}}, \frac{\sqrt{3}}{3})$ | $r \in [.518, .577]$ | $N_n(r) \leq \lceil \frac{2n}{3} \rceil$ |
| Example 2 | Interval **F** | $r \in (0, \frac{1}{2})$ | $r \in (0, .5)$ | $N_n(r) \leq \lceil \frac{n}{\pi} \sin^{-1}(2r) \rceil$ |
| Example 3 | Interval **G** | $r \in [\frac{1}{2}, \frac{\sqrt{3}-1}{\sqrt{2}})$ | $r \in [.5, .518]$ | $N_n(r) \leq \lceil \frac{3n}{5} \rceil$ |
| Lemma 1 | Interval **H** | $r \in [\sqrt{\frac{13}{48}}, \frac{\sqrt{3}}{3})$ | $r \in [.52, .577]$ | $N_n(r) \geq \lceil \frac{n}{2} \rceil$ |
| Figure 2 | Intervals **I** | $r \in [.209, .433]$ for bounds on $N_n(r)$ see * | | |

*

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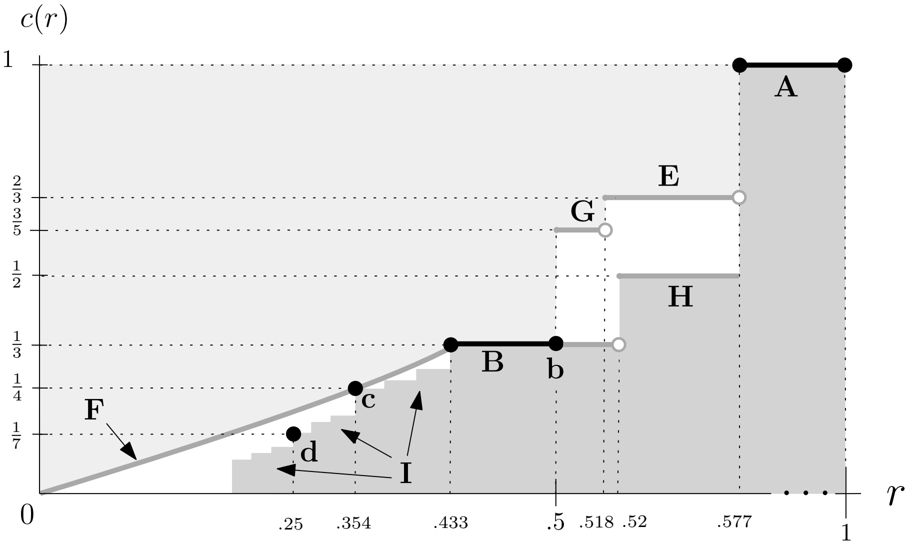

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*$N_n(\frac{1}{4}) = \lceil \frac{n}{7} \rceil$ for $n \neq 7$ and $N_7(\frac{1}{4}) = 2$. Moreover, $c(\frac{1}{4}) = \frac{1}{7}$.*

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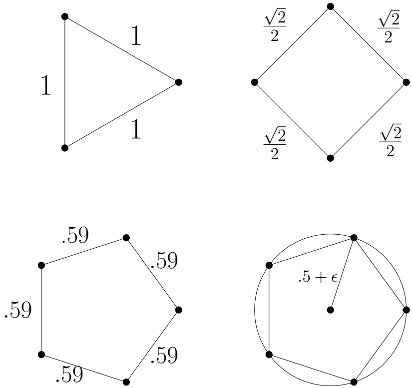

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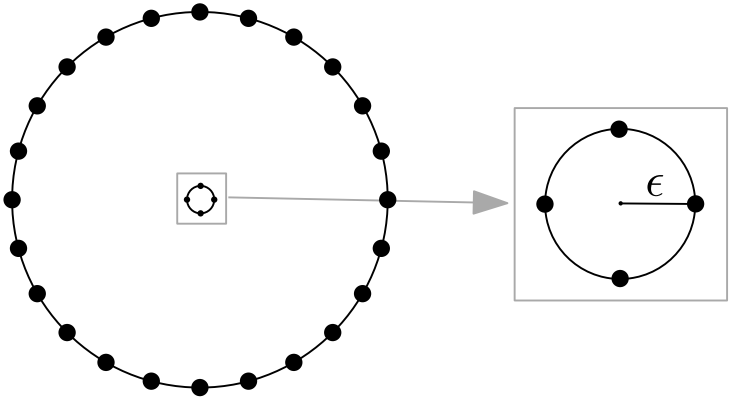

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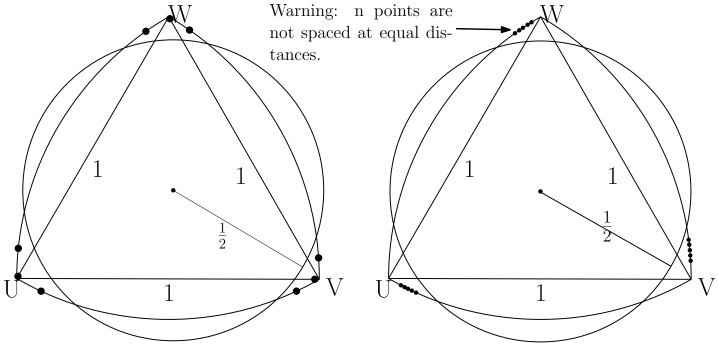


Warning: n points are not spaced at equal distances.

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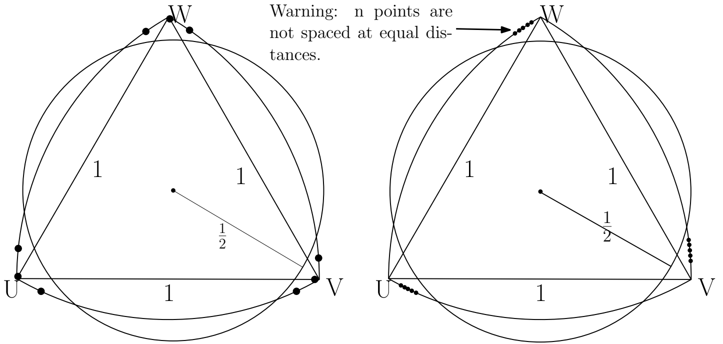

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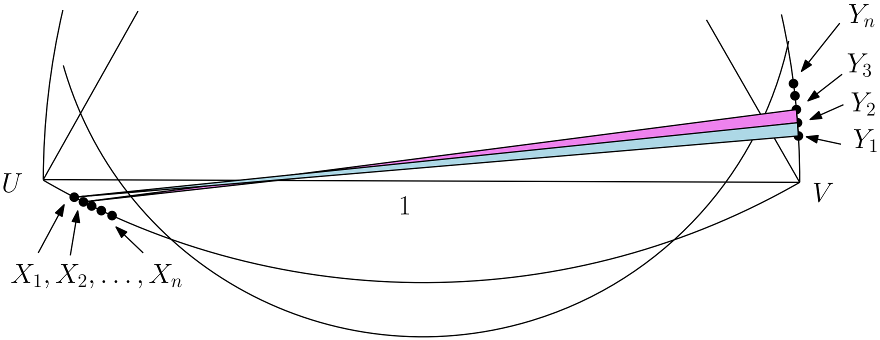

# A note on Borsuk's Theorem

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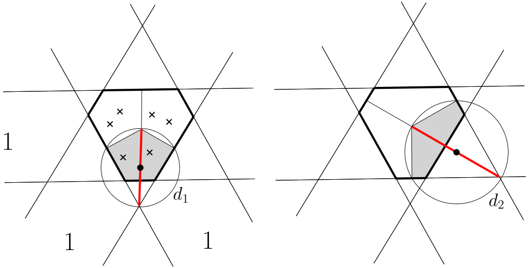
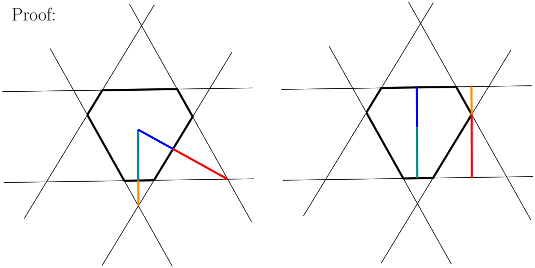
- In 2010 D. Yang, a participant in the Math in Moscow program and found a short elegant proof eliminated continues motion by just computing the diameter of the pentagons formed from dropping the perpendiculars from the shorter sided.
- By dropping the other perpendiculars too we can see a nice proof without words!

# A note on Borsuk's Theorem

Remark on Borsuk's theorem: $d_1 + d_2 = 2$




Proof:



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2. Does the graph of $c(n)$ follow a **step function**?
3. Is there an $r$ such that $N_n(r) = \frac{1}{2}$?
4. Is $N_n(r) = \frac{1}{7}$ for $r \in (\frac{1}{4} - \epsilon, \frac{1}{4} + \epsilon)$ ?

# Thank you

Questions!?