Symmetric convex bodies with the maximal Banach-Mazur distance to the Euclidean ball in dimensions two and three

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John Ellipsoid Theorem

John Ellipsoid Theorem due to Fritz John from 1948 is a fundamental result of convex geometry and functional analysis. It gives an important characterization of an ellipsoid of maximal volume contained in a given symmetric convex body K (called shortly the John ellipsoid of K). The theorem characterizes the John ellipsoid in terms of the contact points of boundaries of K and the John ellipsoid.

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John Ellipsoid Theorem. Let $K \subseteq \mathbb{R}^n$ be a symmetric convex body. Then the Euclidean unit ball $\mathbb{B}^n$ is the maximal volume ellipsoid contained in K if and only if there exist contact points $u^1, u^2, \ldots, u^N \in \mathrm{bd}(K) \cap \mathrm{bd}(\mathbb{B}^n)$ and weights $\lambda_1, \lambda_2, \ldots, \lambda_N > 0$ such that for every $x \in \mathbb{R}^n$ we have

$$x = \sum_{i=1}^{N} \lambda_i \langle x, u^i \rangle u^i.$$

Such a decomposition of the identity is called the John's decomposition. Obviously every symmetric convex body K can be linearly transformed so that $\mathbb{B}^n$ is its John ellipsoid. Moreover, a minimal volume ellipsoid containing K (called Loewner ellipsoid) is also always unique and the John's decomposition exists also for the contact point of this ellipsoid. It is easy to see that the Loewner ellipsoid is a polar (dual) to John ellipsoid.

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Approximation by the John ellipsoid

While the John ellipsoid theorem has many important applications in fields of functional analysis and convex geometry, our point of departure is its classical corollary about the approximation of a convex body by its John ellipsoid.

Corollary of John's Theorem. Let $K \subseteq \mathbb{R}^n$ be a symmetric convex body and let $\mathcal{E} \subseteq K$ be its John ellipsoid. Then $K \subseteq \sqrt{n}\mathcal{E}$.

Proof. By applying a suitable linear transformation we can assume that $\mathcal{E} = \mathbb{B}^n$ is the Euclidean unit ball. It is enough to check that for every $x \in K$ we have $\|x\| \leq \sqrt{n}$.

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Approximation by the John ellipsoid

However, by the John's decomposition applied for $x \in K$ we have

$$\langle x, x \rangle = \sum_{i=1}^{N} \lambda_i \langle x, u^i \rangle^2 \leq \sum_{i=1}^{N} \lambda_i = n,$$

where we have used that $|\langle x, u^i \rangle| \leq 1$ for every $1 \leq i \leq N$, as $u^i \in K^\circ$.

This estimate with $\sqrt{n}$ is fundamental in the theory of Banach-Mazur distance. Originally introduced by Banach in 1932 it serves the purpose of comparing the geometric properties of two normed spaces and quantifies how essentially different the spaces are. For two normed spaces X, Y of dimension n there Banach-Mazur distance is defined as

$$d_{BM}(X,Y) = \inf \|T\| \cdot \|T^{-1}\|,$$

where infimum is taken over all invertible operators $T : X \to Y$. Similarly we can define a Banach-Mazur distance between two symmetric convex bodies $K, L \subseteq \mathbb{R}^n$. In this case

$$d_{BM}(K,L) = \inf\{r > 0 : K \subseteq T(L) \subseteq rK\}.$$

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The estimate with $\sqrt{n}$ reads now simply as $d_{BM}(K, \mathbb{B}^n) \leq \sqrt{n}$. This estimate is fundamental, as for example, when combined with the triangle inequality, it yields that $d_{BM}(K, L) \leq n$ for any two symmetric convex bodies $K, L \subseteq \mathbb{R}^n$. In other words, the diameter of the Banach-Mazur compactum is upper bounded by n. It was later proved by Gluskin by a famous probabilistic construction, that the diameter can be lower bounded by cn for some absolute constant $c > 0$. Thus, the diameter of the symmetric Banach-Mazur compactum is of a linear order.

Characterization of the maximal distance

Because the inequality $d_{BM}(K, \mathbb{B}^n) \leq \sqrt{n}$ for a symmetric convex body $K \subseteq \mathbb{R}^n$ is fundamental, it is very natural to ask, when the equality holds. In other words, what are the symmetric convex bodies with the maximal possible distance to the Euclidean unit ball? Let us remark here, that in case of K being a non-symmetric convex body it is possible to prove (again using the John's decomposition) that $d_{BM}(K, \mathbb{B}^n) \leq n$. In this case, it is known that if the equality holds, then K has to be a simplex in $\mathbb{R}^n$. Moreover, this uniqueness of simplex is stable , that is if $d_{BM}(K, \mathbb{B}^n) \geq n - \varepsilon$, then $d_{BM}(K, S_n) \leq 1 + c\varepsilon^2$ for some constant $c > 0$.

It turns out that in the symmetric case, the situation is more complicated. It is quite easy to prove that the n-dimensional cross-polytope and the n-dimensional parallelotope (so the unit balls of ℓ_1^n and ℓ_∞^n respectively) have the distance to the Euclidean unit ball equal to $\sqrt{n}$. But are they the only ones?

Characterization of the maximal distance

Generally, they are not the only ones. In every dimension $n \geq 4$ there are examples of n-dimensional symmetric convex bodies with the distance equal to $\sqrt{n}$, which are not linearly equivalent to a parallelotope or a cross-polytope. However, it was proved by Milman and Wolfson that any normed space with the maximal distance has a subspace of a roughly logarithmic dimension, which is isometric to ℓ_1.

Therefore it is natural to ask, what happens then in the dimensions 2 and 3? While there are several papers related to the spaces with the maximal distance to the Euclidean space, it is surprisingly hard to find any information about these two cases. It turns out that in those dimensions, the maximizers are only the obvious one. While the two-dimensional case can be regarded as a simple exercise, the three-dimensional case is much more challenging. There exists a strong indication that the proof of this fact was known to some mathematicians, but it seems that the result was never published.

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Historical background

While the two-dimensional characterization can be regarded as rather simple with the current state of knowledge, already this case is not easily established in the literature. It can be traced back to independent works of John from 1936 and Behrend from 1937 (the latter available only in German), where it is proved that $d_{BM}(K, \mathbb{B}^2) \leq d_{BM}(\mathcal{P}^2, \mathbb{B}^2) = \sqrt{2}$ for any symmetric convex body $K \subseteq \mathbb{R}^2$, with equality if and only if K is a parallelogram. Neither of them used the language of Banach-Mazur distances and their proofs are somewhat convoluted from a modern point of view. Their works preceded the John Ellipsoid Theorem, though during the process of, Behrend has basically established the existence and uniqueness of the John ellipse in the symmetric planar case along with the estimate of $\sqrt{2}$.

Historical background

It was noted also much later by Lewis in 1979 that the two-dimensional case is a consequence of a more general result about Banach ideal norms. Lewis attributed this observation to Figiel and Davis, so seemingly, he was not aware of the previous works of John and Behrend. The three discussed papers seem to be the only ones where some argument for the two-dimensional case is given.

Concerning the three dimensional case, to our best knowledge, the only information that can be found in the existing literature is the following passage from the paper *Structure of normed spaces with extremal distance to the Euclidean space* of Anisca, Tcaciuc, and Tomczak-Jaegermann from 2005:

"Some further properties of spaces with the maximal Euclidean distance were known at the beginning of the 1990's to several people working in the area (Arias, Komorowski, Maurey and Tomczak-Jaegermann). In particular they showed that spaces with the maximal Euclidean distance have a unique distance ellipsoid, and that the only 3-dimensional spaces with the maximal distance are the obvious ones, $X = \ell_1^3$ and $X = \ell_\infty^3$."

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Unfortunately, no proof, reference, or even suggestion how to approach proving such a fact is provided there. Furthermore, the authors refer to another unpublished result, reportedly proved by Maurey, which states in particular that if $d_{BM}(K, \mathbb{B}^n) = \sqrt{n}$, then the distance ellipsoid of K is unique. With such a result at hand, it would be quite simple to prove that indeed $\mathcal{P}^3$ and $\mathcal{C}^3$ are, up to affine transformation, the only symmetric convex bodies in $\mathbb{R}^3$ with the distance $\sqrt{3}$ to $\mathbb{B}^3$.

As almost two decades have passed since the paper of Anisca, Tcaciuc, and Tomczak-Jaegermann, we revisit the forgotten case of $n = 3$. We were able to confirm the claim from that paper, without relying on any unpublished results. We actually do not use anything else than the basic John Ellipsoid Theorem (in terms of referring to some external results).

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Mean of ellipsoids

One of the main ingredients is the lemma about combining different distance ellipsoids, which can generally reduce the contact points.

Lemma about means of ellipsoids. Let K be a symmetric convex body such that $\mathbb{B}^n \subseteq K \subseteq d\mathbb{B}^n$ for some $d \geq 1$. Moreover, suppose that vectors $v^1, \ldots, v^n \in \mathbb{R}^n$ form an orthonormal basis, $\alpha_1, \ldots, \alpha_n > 0$ are reals, $\lambda \in [0, 1]$ is a real parameter and the ellipsoid $E_\lambda \subseteq \mathbb{R}^n$ is defined as

$$E_\lambda = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{n} \frac{\langle x, v^i \rangle^2}{\alpha_i^{2\lambda}} \leq 1 \right\}.$$

Let $V \subseteq \mathbb{R}^n$ be a linear subspace spanned by all vectors v^i such that $\alpha_i = 1$. If $E_1 \subseteq K \subseteq dE_1$, then for every $\lambda \in (0, 1)$ we have

(i) $E_\lambda \subseteq K \subseteq dE_\lambda$,

(ii) $\mathrm{bd}(K) \cap \mathrm{bd}(dE_\lambda) \subseteq V$,

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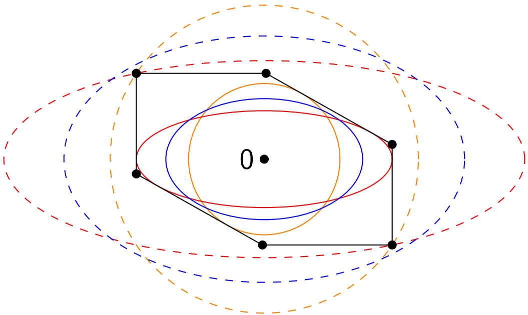

Figure: K (black), $\mathbb{B}^2$ (orange, solid), E (red, solid), $E_{\frac{1}{2}}$ (blue, solid). The dashed ellipses are obtained from the solid ellipses by scaling with factor $d \approx 2$. Neither of the principal semi-axes of E has length 1, so $\mathrm{bd}(K)$ is guaranteed to not intersect $\mathrm{bd}(E_\lambda)$ and $\mathrm{bd}(dE_\lambda)$ for any $\lambda \in (0, 1)$.

Main idea of the proof of the three-dimensional case

By applying a suitable linear transformation we can assume that $\mathbb{B}^3 \subseteq K$ is the John ellipsoid for K. Let us also take an ellipsoid $E \subseteq \mathbb{R}^3$ such that $\sqrt{3}E$ is the Loewner ellipsoid of K. A classical argument shows that $E \subseteq K$ (as the John's decomposition exists also for Loewner ellipsoid) and hence $E \subseteq K \subseteq \sqrt{3}E$. Let us write E in the form

$$E = \left\{ x \in \mathbb{R}^3 : \sum_{i=1}^{3} \frac{\langle x, v_i \rangle^2}{\alpha_i^2} \leq 1 \right\},$$

where v^1, v^2, v^3 is an orthonormal basis of $\mathbb{R}^3$ and $\alpha_1, \alpha_2, \alpha_3 > 0$. By Lemma 1, the ellipsoid

$$F = \left\{ x \in \mathbb{R}^3 : \sum_{i=1}^{3} \frac{\langle x, v_i \rangle^2}{\alpha_i} \leq 1 \right\}$$

satisfies $F \subseteq K \subseteq \sqrt{3}F$ and $\mathrm{bd}(K) \cap (\mathrm{bd}(F) \cup \mathrm{bd}(\sqrt{3}F)) \subseteq V$, where $V \subseteq \mathbb{R}^3$ is a linear subspace spanned by all vectors v^i such that $\alpha_i = 1$.

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Main idea of the proof of the three-dimensional case

We shall proceed by considering cases based on $\dim V$.

The case $\dim V \leq 1$ is trivial, as in this case $\mathrm{bd}(K) \cap \mathrm{bd}(F) = \emptyset$ or $\mathrm{bd}(K) \cap \mathrm{bd}(\sqrt{3}F) = \emptyset$, as K cannot have a common boundary point with both F and $\sqrt{n}F$ within the same line through the origin. Thus either $F \subseteq \mathrm{int}\, K$ or $K \subseteq \mathrm{int}\, \sqrt{3}F$, which gives us a contradiction with the assumption $d_{BM}(K, \mathbb{B}^3) = \sqrt{3}$.

The case $\dim V = 3$ is easy and it is a simple case work. In this case $E = \mathbb{B}^3$, so the John and Loewner ellipsoids are homothetic. In particular, there exists a John's decomposition for the contact points of K and $\mathbb{B}^3$, but also for the contact points of K and $\sqrt{3}\mathbb{B}^3$. Having these two John decompositions at hand it is relatively easy to get that K must be a cube or an octahedron.

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The main work to do is therefore in the case $\dim V = 2$. In this case the crucial observation is the fact that there must exists two contact points $x, y \in \mathrm{bd}(K) \cap \mathrm{bd}(\sqrt{3}\mathbb{B}^3) \cap V$ (such that $x \neq \pm y$) and that in this case we have $|\langle x, y \rangle| \leq 1$. This allows to construct a linear perturbation of K, for which the distance of $\sqrt{3}$ to $\mathbb{B}^3$ is decreased.

Interestingly, this upper estimate of the inner-product of contact points by 1 seems to be a threshold, for which this idea of linear perturbation starts to work. And this is exactly what can be deduced. In the n-dimensional case the upper bound would be $n-2$ and it is optimal as the case of the n-dimensional cube shows.

The main work to do is therefore in the case $\dim V = 2$. In this case the crucial observation is the fact that there must exists two contact points $x, y \in \mathrm{bd}(K) \cap \mathrm{bd}(\sqrt{3}\mathbb{B}^3) \cap V$ (such that $x \neq \pm y$) and that in this case we have $|\langle x, y\rangle| \leq 1$. This allows to construct a linear perturbation of K, for which the distance of $\sqrt{3}$ to $\mathbb{B}^3$ is decreased.

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A counterexample in dimensions $n \geq 4$

Generally the counterexamples for $n \geq 4$ are known, although it is rather difficult to establish in the literature an explicit example that works for all dimensions $n \geq 4$. One easy example, that covers all these dimensions is a symmetric convex body arises from the n-dimensional cube by cutting a symmetric pair of vertices by hyperplane (and a symmetric one). Clearly the cube has the John decomposition from $\mathbb{B}^n$ supported on centers of facets (as $\mathbb{B}^n$ is the John ellipsoid), but it has also another one supported on the vertices (as $\sqrt{n}\mathbb{B}^n$ is the Loewner ellipsoid). If $n \geq 4$ then we can throw out one pair of vertices and the John decomposition still persists. It means that $\mathbb{B}^n$ and $\sqrt{n}\mathbb{B}^n$ are still John/Loewner ellipsoids for this modified cube. Now it is very easy to prove that if John/Loewner ellipsoids of a given convex body are homothetic with ratio $\sqrt{n}$, then the Banach-Mazur distance is equal to $\sqrt{n}$.

Stability of the parallelogram

Knowing that the parallelogram is the unique planar, symmetric convex body with the distance $\sqrt{2}$ to the Euclidean unit disc, it is natural to ask about the stability of the parallelogram. We have the following result.

Stability Theorem. Let $\varepsilon > 0$ and let $K \subseteq \mathbb{R}^2$ be a symmetric convex body such that $d_{BM}(K, \mathbb{B}^2) \geq \sqrt{2} - \varepsilon$. Then

$$d_{BM}(K, \mathcal{P}^2) \leq 1 + c\varepsilon,$$

where $c = \frac{10}{\sqrt{2}} \approx 7.07$.

Obviously the linear order in this estimate is best possible, as if K is any convex body with $d_{BM}(K, \mathbb{B}^2) = \sqrt{2} - \varepsilon$, then by the triangle inequality we obviously have $d_{BM}(K, \mathcal{P}^2) \geq \frac{\sqrt{2}}{\sqrt{2}-\varepsilon} \geq 1 + \frac{\varepsilon}{\sqrt{2}}$.

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Fun corollary

As a consequence we obtain the following.

Corollary.For every symmetric convex body $K \subseteq \mathbb{R}^2$ we have $d_{BM}(K, \mathbb{B}^2) < 1.363$ or $d_{BM}(K, \mathcal{P}^2) < 1.363$.

It is interesting to note that the problem about covering the symmetric Banach-Mazur compactum with balls centered at $\mathbb{B}^2$ and $\mathcal{P}^2$ has been proposed during the open problem session of the workshop "Interplay between Geometric Analysis and Discrete Geometry" that was held in 2023 in Mexico.

On the other hand, the unit ball of ℓ_4^4 is of distance $\sqrt[4]{2} \approx 1.189$ to both $\mathbb{B}^2$ and $\mathcal{P}^2$.

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The end

Thank you for your attention!