Compositions of Sets in Geometric Ramsey Theory

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Ramsey Sets

For a set $X \subset \mathbb{R}^d$, the chromatic number $\chi(\mathbb{R}^n, X)$ is the minimum number of colors to color points of $\mathbb{R}^n$ without a monochromatic isometric copy of X.

A set is Ramsey if $\chi(\mathbb{R}^n, X) \to \infty$ as $n \to \infty$

Erdős, Graham, Montgomery, Rothschild, Spencer, Straus, 1973: A set is Ramsey only if it is finite and spherical

Question (EGMRSS)/Conjecture (Graham):

Should it be iff?

Known Ramsey sets: pair of points, triangles, non-degenerate simplices, direct products of Ramsey sets, dodecahedron, icosahedron, 120-cell etc.

Exponentially Ramsey Sets

A set X is exponentially Ramsey if $\chi(\mathbb{R}^n, X) = (c + o(1))^n$ for some $c > 1$.

What is the best c for different sets?

Raigorodskii, 2000: $\chi(\mathbb{R}^n, X) \geqslant (\psi + o(1))^n$ for a two-point X and $\psi = 1.239...$

The case of unit triangles

Frankl and Rödl, 1987: unit triangles are exponentially Ramsey

Sagdeev, 2019: $\chi(\mathbb{R}^n, \triangle) \geqslant (1.0140... + o(1))^n$

Naslund, 2020: $\chi(\mathbb{R}^n, \triangle) \geqslant (1.0144... + o(1))^n$

Theorem (AK, Sagdeev, Zakharov, 2023)

$$\chi(\mathbb{R}^n, \triangle) \geqslant (\psi^{1/2} + o(1))^n = (1.0742... + o(1))^n$$

Exponentially Ramsey Sets

A set X is super-Ramsey if there exist $C, c > 1$ s.t. for any n there is a set V of size at most C^n s.t. $|V|/\alpha(V, X) > (c + o(1))^n$.

Theorem(Frankl and Rödl, 1990)

If X_1 and X_2 are super-Ramsey, then so is $X_1 \times X_2$.

How to get a bound for a unit simplex? via $\triangle^k \subset \square^{k+1}$

'Original' Frankl and Rödl gives $\chi(\mathbb{R}^n, \triangle^k) \geqslant \left(1 + 2^{-2^{k+4}} + o(1)\right)^n.$

'Product' Frankl and Rödl gives $\chi(\mathbb{R}^n, \triangle^k) \geqslant \left(1 + \frac{1}{(k+1)^2 2^{k+1}} + o(1)\right)^n.$

Theorem (AK, Sagdeev, Zakharov, 2023)

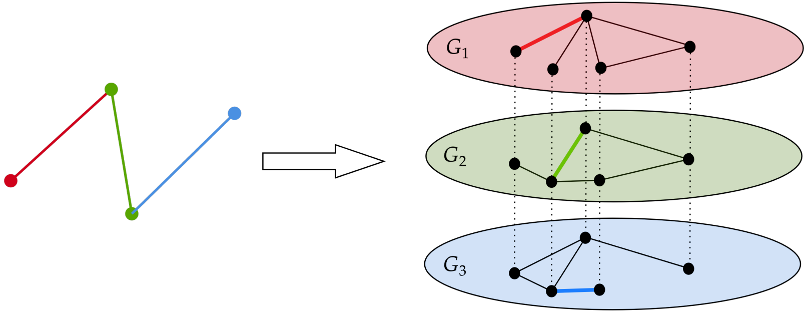
$$\chi(\mathbb{R}^n, \triangle^k) \geqslant \left(\psi^{1/(k+1)} + o(1)\right)^n \geqslant \left(1 + \frac{\psi}{k+1} + o(1)\right)^n$$

Our approach: Tree-like concatenation

Lemma (Rainbow trees in large vertex sets)

$G_i = (V, E_i)$, $i \in [k]$ are graphs on the same vertex set. Fix a tree T with k ordered edges. Then in any $W \subset V$, $|W| > \alpha(G_1) + \ldots + \alpha(G_k)$, there exists a homomorphic copy of T in $G = (V, E_1 \cup \ldots \cup E_k)$ s.t. the image of the i-th edge belongs to E_i.

Proof: simple induction on the size of the tree.



Orthogonal trees

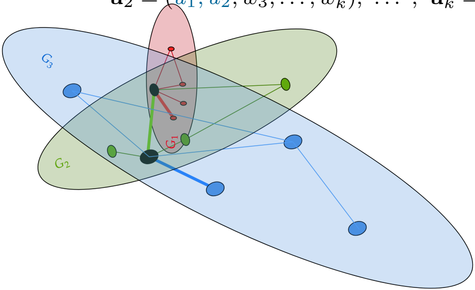
$G_i' = (V_i, E_i')$, $i \in [k]$, be a family of k graphs, and $(u_i, w_i) \in E_i'$, $i \in [k]$.

An orthogonal star:

$$\mathbf{w}_0 = (w_1, \ldots, w_k),\ \mathbf{w}_1 = (u_1, w_2, \ldots, w_k),\ \ldots\ ,\ \mathbf{w}_k = (w_1, \ldots, w_{k-1}, u_k).$$

An orthogonal path:

$$\mathbf{u}_0 = (w_1, \ldots, w_k),\ \mathbf{u}_1 = (u_1, w_2, \ldots, w_k),$$
$$\mathbf{u}_2 = (u_1, u_2, w_3, \ldots, w_k),\ \ldots\ ,\ \mathbf{u}_k = (u_1, \ldots, u_k)$$



Orthogonal trees

$G_i = (V_1 \times \cdots \times V_k, E_i)$, $E_i = \{(\mathbf{x}, \mathbf{y}) : (x_i, y_i) \in E'_i, x_j = y_j \text{ for } j \neq i\}$

Lemma (Avoiding orthogonal stars/paths)

If $W \subset V_1 \times \cdots \times V_k$ contains no orthogonal stars (or no orthogonal paths), then

$$\frac{|W|}{|V_1 \times \cdots \times V_k|} \leqslant \sum_{i=1}^{k} \frac{\alpha(G_i)}{|V_i|}.$$

Application to $\chi(\mathbb{R}^n, \triangle)$

k-semicross SC^k: $\{\mathbf{0}, \mathbf{e}_1, \ldots, \mathbf{e}_k\} \subset \mathbb{R}^k$, where $\{\mathbf{e}_i\}$ is the standard basis

For $V \subset \mathbb{R}^n$, consider a unit distance graph $G = (V, E)$: two points of V are connected iff they are at unit distance apart.

An orthogonal star in the Cartesian power V^k is isometric to SC^k. Thus

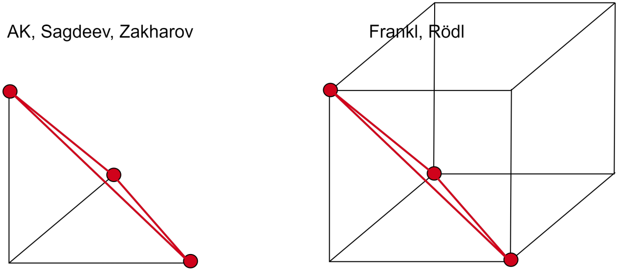
$$\alpha\big(V^k, \mathrm{SC}^k\big) \leqslant k|V|^{k-1}\alpha(V).$$

Substitute in the Lemma the set $A \subset \mathbb{R}^n$ giving the best bound for the ratio $|A|/\alpha(A)$ (where α is w.r.t. avoiding unit distances):

$$\chi\big(\mathbb{R}^{kn}, \mathrm{SC}^k\big) \geqslant \frac{|A|^k}{\alpha\big(A^k, \mathrm{SC}^k\big)} \geqslant \frac{|A|^k}{k|A|^{k-1}\alpha(A)} \geqslant \big(\psi_2 + o(1)\big)^n.$$

Application to $\chi(\mathbb{R}^n, \triangle)$

Embed $\triangle^k \subset \mathrm{SC}^k$, instead of $\triangle^k \subset \square^{k+1}$:



$\chi(\mathbb{R}^n, \triangle^k)$ for $k = k(n)$

Best lower bound, Zakharov, 2023:

$$\chi\big(\mathbb{R}^n, \triangle^k\big) \geqslant ce^{c\sqrt{n/k}} \quad \text{for some } c > 0$$

Best upper bound, Prosanov, 2018:

$$\chi\big(\mathbb{R}^n, \triangle^k\big) \leqslant \Big(1 + \sqrt{2(k+1)/k} + o(1)\Big)^n$$

Question
For $\varepsilon > 0$, is there $k = k(\varepsilon)$ such that

$$\chi\big(\mathbb{R}^n, \triangle^k\big) \leqslant \big(1 + \varepsilon + o(1)\big)^n \text{ as } n \to \infty?$$

Weak sunflowers

A collection of $k \geqslant 3$ sets is called a weak k-sunflower if all their pairwise intersections are of the same cardinality.

$G_k(n)$: max size of a family $\mathcal{F} \subset 2^{[n]}$ with no weak k-sunflowers.

Kostochka, Rödl, 1998: $G_k(n) \geqslant k^{c(n \log n)^{1/3}}$

Frankl, Rödl, 1987: $G_k(n) \leqslant \big(2 - \varepsilon_k + o(1)\big)^n$

Naslund, 2022: $G_3(n) \leqslant \big(1.837 + o(1)\big)^n$

Theorem (AK, Sagdeev, Zakharov, 2023)

$$G_k \leqslant \big(2\psi^{-1/k} + o(1)\big)^n$$

as $n \to \infty$, where $\psi = \frac{1+\sqrt{2}}{2} = 1.207...$

For $k = 3$, is worse: $G_3(n) \leqslant \big(1.879 + o(1)\big)^n$, but better for $k \geqslant 4$.

Frankl–Rödl from Frankl-Wilson

Theorem (Frankl and Wilson, 1981)

If a family $\mathcal{F} \subset \binom{[n]}{k}$ is t-avoiding (i.e., no $|F_1 \cap F_2| = t$), $t < k/2$ and $k - t$ is a prime power, then $|\mathcal{F}| \leqslant \sum_{i=0}^{k-t-1} \binom{n}{i}$.

Theorem (Frankl and Rödl, 1987)

If a family $\mathcal{F} \subset \binom{[n]}{k}$ is t-avoiding and $t < k/2$, then $|\mathcal{F}| \leqslant (2-\epsilon)^n$.

Specific bounds from Frankl–Wilson are essentially sharp, and from Frankl–Rödl are quite bad.

Keevash and Long, 2017: Frankl–Rödl from Frankl-Wilson using that any number is a sum of 4 primes and dependent random choice.

Using orthogonal path-like concatenation, we can get a much shorter and more efficient reduction. Decompose $n = \sum_{i=1}^{4} n_i$, where $n_i \sim n/4$ and $t = \sum_{i=1}^{4} t_i$, where $t_i \sim t/4$ so that $n_i - t_i$ is a prime, and apply Frankl-Wilson in each piece.

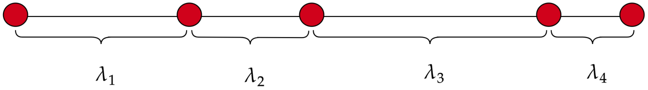
Max-norm Ramsey Theory

For a set $X \subset \mathbb{R}^d$, the chromatic number $\chi(\mathbb{R}^n_\infty, X)$ is the minimum number of colors to color points of $\mathbb{R}^n_\infty$ without a monochromatic isometric copy of X.

Theorem (Kupavskii, Sagdeev, 2021)
Any finite metric space X is exponentially ℓ_∞-Ramsey.

One-dimensional metric spaces (batons)

Given $\lambda_1, \ldots, \lambda_k > 0$, set $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k)$. For all $i \in \{0, \ldots, k\}$, define $\sigma_i = \sum_{j=1}^{i} \lambda_j$. The set $\{\sigma_0, \ldots, \sigma_k\} \subset \mathbb{R}$ a *baton* $\mathcal{B}(\boldsymbol{\lambda})$. For $\mu > 0$, put $\mathcal{B}_k(\mu) = \mathcal{B}((\underbrace{\mu, \ldots, \mu}_{k}))$



Max-norm Ramsey Theory

For a subset $S \subset \mathbb{Z}$: $d(\mathbb{Z}, S)$ is the supremum of upper densities of $A \subset \mathbb{Z}$ s.t. for all $x \in \mathbb{Z}$, A contains neither $S + x$ nor $-S + x$.

Theorem (Frankl, Kupavskii, Sagdeev)

Let $k \in \mathbb{N}$ and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k)$, $\lambda_i > 0$. Then

$$\chi(\mathbb{R}^n_\infty, \mathcal{B}) = \left(d(\mathbb{Z}, \mathcal{B})^{-1} + o(1)\right)^n.$$

For general batons, the situation is more complicated, but we can prove

Theorem (Frankl, Kupavskii, Sagdeev)

If $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k)$, $\lambda_i > 0$ and are linearly independent over $\mathbb{Z}$, then

$$\chi(\mathbb{R}^n_\infty, \mathcal{B}(\boldsymbol{\lambda})) = \left(\frac{k+1}{k} + o(1)\right)^n.$$

Products of 1-dimensional spaces

Theorem (Frankl, AK, Sagdeev)

Let $k, m \in \mathbb{N}$ and $\lambda_1, \ldots, \lambda_m$ be positive reals. Then

$$\chi\big(\mathbb{R}^n_\infty, \mathcal{B}_k(\lambda_1) \times \cdots \times \mathcal{B}_k(\lambda_m)\big) = \left(\frac{k+1}{k} + o(1)\right)^n.$$

Open problems

Problem (Frankl, AK, Sagdeev)

Let $\mathcal{Y}_1, \mathcal{Y}_2$ be two arbitrary one-dimensional metric spaces and c_1, c_2 be positive reals such that $\chi(\mathbb{R}^n_\infty, \mathcal{Y}_i) = (c_i + o(1))^n$, $i = 1, 2$. Set $c = \min\{c_1, c_2\}$. Is it always true that

$$\chi(\mathbb{R}^n_\infty, \mathcal{Y}_1 \times \mathcal{Y}_2) \geqslant (c + o(1))^n \,?$$

Problem (Frankl, AK, Sagdeev)

Is there an infinite set B with $\chi(\mathbb{R}^n_\infty, B) = n + 1$?

Problem (AK, Sagdeev, Zakharov)

Understand the behaviour of $\chi(\mathbb{R}^n, \triangle^k)$ for $k = k(n)$.