Linear programming bounds for problems in discrete geometry

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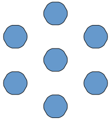


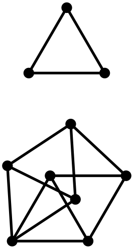
Overview

- Unit distance graphs: basic notions and problem formulation
- Hierarchy of linear programs I:fractional chromatic number(FCN)
- Hierarchy of linear programs II: the geometric fractional chromatic number (GFCN)
- Computer search: an example for GFCN=4
- Hierarchy of linear programs III: Fourier analysis
- Computer search: a witness graph for density bounds

Joint with G. Ambrus, A. Csiszárik, I. Ruzsa, D. Varga, P. Zsámboki
(the talk does not follow the historical order of our results)

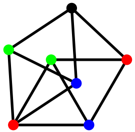
Unit distance graphs

- In this talk, everything happens in the Euclidean plane. (Higher dimensions also possible, but not today!)
- A unit distance graph (UDG) is a finite subset of the plane, where two vertices are connected if and only if they are unit distance away.



Chromatic number, independence ratio

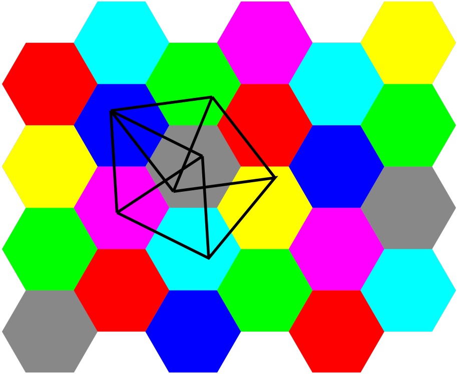
- Chromatic number (CN) of a unit distance graph G: the minimal number of colours needed if adjacent vertices must have different colour.
- Independence ratio (IR): the relative size of the largest independent set in G



- For the Moser spindle we have $CN(Moser) = 4$, $IR(Moser) = 2/7$.
- $CN(G) \geq \frac{1}{IR(G)}$

Hadwiger-Nelson problem

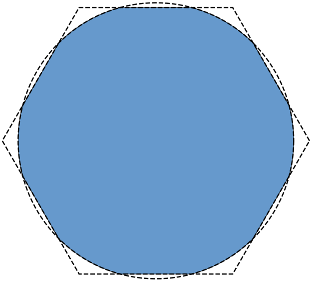
- What is the chromatic number of the whole plane? At most 7, by the picture.
- The Moser spindle proves that it is at least 4. Brothers Leo and William Moser found it in 1961.
- Aubrey de Grey (2018): The chromatic number of $\mathbb{R}^2$ is at least 5 (a graph on 1581 vertices)

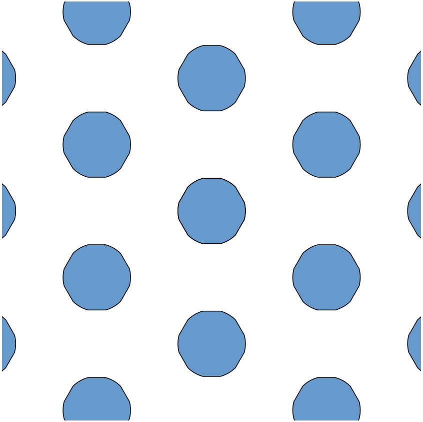


1-avoiding sets

- A set is called 1-avoiding, if there are no two points unit distance away.
- In 1966, Leo Moser asked for the highest density of a measurable 1-avoiding subset A of the plane.
- More exactly, he defined $m_1(\mathbb{R}^2)$ to be the supremum of the upper densities of 1-avoiding measurable sets in $\mathbb{R}^2$, and asked about its value.
- Easy: we can assume that A is periodic with ε loss in the density.

Croft's tortoise: $m_1(\mathbb{R}^2) > 0.2293$





Erdős' conjecture: $m_1(\mathbb{R}^2) < 0.25$

- The Croft construction might be optimal (people have tried and failed to improve it since its introduction in 1967.)
- Paul Erdős formulated a weaker conjecture:

Erdős' Conjecture (1985)

$m_1(\mathbb{R}^2) < 1/4$. That is, the supremum of the upper densities of 1-avoiding measurable sets in $\mathbb{R}^2$ is less than $1/4$.

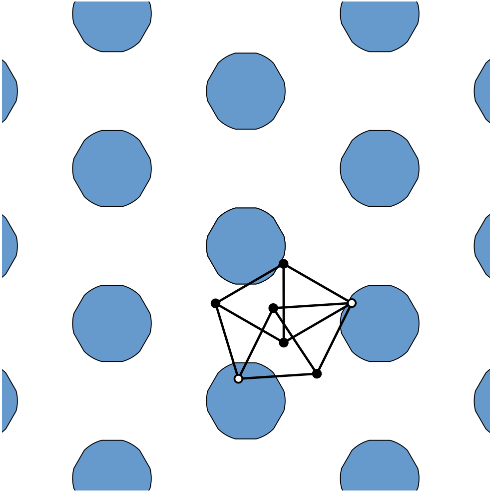
Remark: this implies that at least 5 colours are needed for the plane, if we use measurable colour classes.

Linear programs induced by 1-avoiding sets

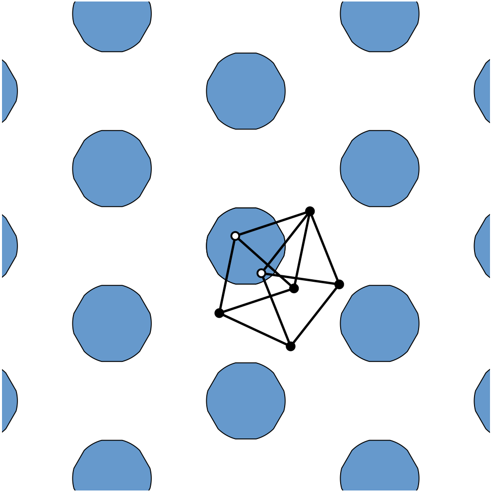
We want to prove that all 1-avoiding sets have low densities.

- My opponent brings a 1-avoiding set A, and claims that it has high density.
- I bring a small unit distance graph X (this will be my witness, that he is wrong).
- I randomly drop X on the plane, and take its intersection with A.
- I record the relative frequencies of the intersections. This will give a probability distribution on the independent sets of X.
- We write up linear constraints on the frequencies.

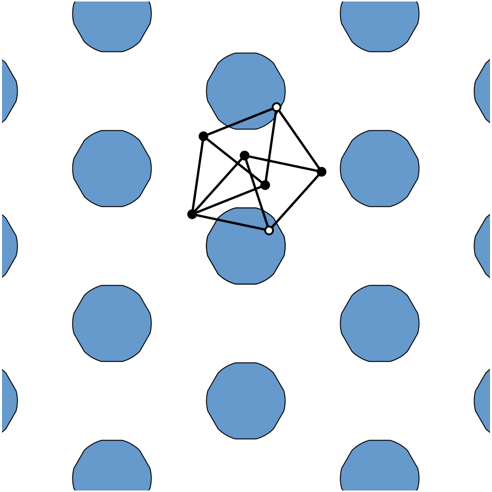
Randomly dropping a graph on the set



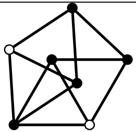
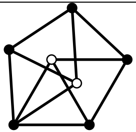
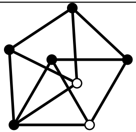
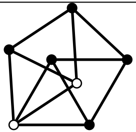
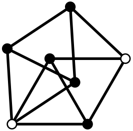
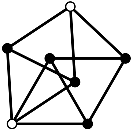
Randomly dropping a graph on the set



Randomly dropping a graph on the set



Histogram of intersection patterns

	10%		3%
	11%		0%
	15%		15%
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Linear program I: fractional chromatic number

- Let δ denote the density of the 1-avoding set A, and $a(I)$ denote the occurrence rate corresponding to an independent set I of X in the histogram.
- We can write up linear constraints for the "atomic variables" $a(I)$.
- $a(I) \geq 0$, $\sum_I a_I = 1$, and for every $z \in X$ we have $\sum_{z \in I} a(I) = \delta$.
- We get an upper bound on the density δ by solving an LP: maximize δ with the above constraints.
- The reciprocal of the maximal value is called the *fractional chromatic number* $FCN(X)$. (Divide everything by δ and minimize $\sum_I a(I)$.)

FCN evolution

To prove $\delta < 1/4$ we need a graph X with $FCN(X) > 4$. The evolution of the best know values of FCN:

- 3.5 (Moser 1966)
- 3.55 (Fisher and Ullman 1997)
- 3.61 (Cranston and Rabern 2015)
- 3.89 (Bellitto, Pecher, and Sedillot 2018)
- 3.97 (Parts 2019, unpublished)
- 3.99 (Parts 2020, unpublished)

The graphs are getting pretty large in the last three cases. Maybe $FCN(\mathbb{R}^2) = 4???$

Linear program II: geometric fractional chromatic number

There are further linear constraints on the variables $a(I)$, induced by geometric congruencies.

If $K \cong J$ are congruent independent sets of X, then $\sum_{K \subset I} a(I) = \sum_{J \subset I} a(I)$

We define the geometric chromatic number $GFCN(X)$ by adding these constraints.

To prove $\delta < 1/4$ we need a graph X with $GFCN(X) > 4$.

Theorems for FCN, GFCN and IR

- A fairly trivial monotonicity relation holds: if $X \subset Y$ then $FCN(X) \leq FCN(Y)$ and $GFCN(X) \leq GFCN(Y)$.
- Also, we have $\frac{1}{IR(X)} \leq FCN(X) \leq GFCN(X)$ for every finite X.
- Interestingly, the values are equal in the limit:

$$\frac{1}{IR(\mathbb{R}^2)} = FCN(\mathbb{R}^2) = GFCN(\mathbb{R}^2).$$

From GFCN to FCN: blow-up construction

Idea of the proof of $FCN(\mathbb{R}^2) = GFCN(\mathbb{R}^2)$

- Assume that for a graph X the value $GFCN(X) = \gamma$ is large.
- Define a huge graph Y by pasting many translated and rotated copies of X on the plane.
- Due to the fact that Y is "almost" invariant under translations and rotations, $FCN(Y)$ will be almost as large as γ.

(The proof works only in dimension 2.)

Computer search: $GFCN(X) = 4$

We want to find a graph with "large" FCN or GFCN. Although there is no theoretical advantage in considering GFCN over FCN, there is a huge practical advantage. The graphs are smaller!

GFCN

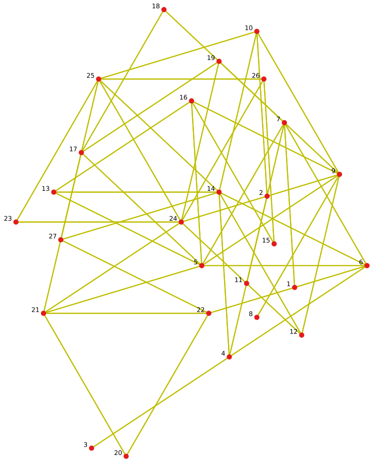
There exists a graph X on 27 vertices such that $GFCN(X) = 4$.

(Curiosity: we only find "ugly" solutions of the dual LP.)

Corollary: The (upper) density $\delta \leq 1/4$ for all 1-avoiding sets.

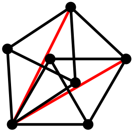
But this still falls short of proving Erdős' conjecture: $\delta < 1/4$.

The result of the beam search: X_{27}



Linear program III: Fourier analysis

By the geometric constraint, if two segments $L_1, L_2 \subset X$ have the same length, then $\sum_{L_1 \subset I} a(I) = \sum_{L_2 \subset I} a(I)$.



This defines an "autocorrelation" function $f(x)$ for the values $x = |L_1| = |L_2|$ appearing as distances in the graph X.
The value $f(x)$ can be defined for any x as the average value of $\delta(A \cap (A + v))$ where $v \in \mathbb{R}^2$ is a vector of length x.
This function f can then be expanded as $f(x) = \sum_{t \geq 0} \kappa(t) \Omega_2(tx)$ where Ω_2 is a Bessel function, and the "Fourier coefficients" $\kappa(t)$ are nonnegative. This leads to new linear constraints.

The final LP

Maximize $\sum_{t\geq 0}\kappa(t)$ subject to

(CP) $\kappa(t)\geq 0$ for every $t\geq 0$

(IEP) $a(I)\geq 0$

(C0) $\sum_{t\geq 0}\kappa(t)\Omega_2(t)=0$ $\Leftarrow$ 1-avoiding set

(IET) $\sum_I a(I)=1$

(IE1) $\sum_{t\geq 0}\kappa(t)-\sum_{z\in I)} a(I)=0$ for every $z\in X$

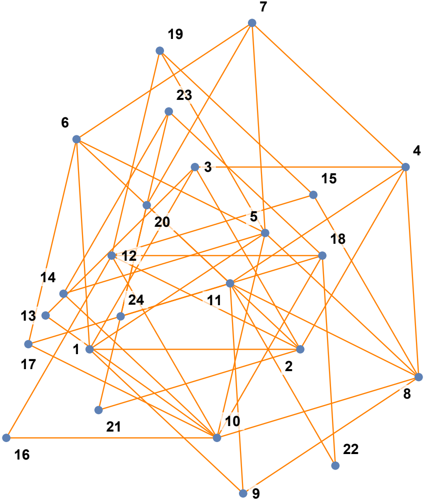
(IE2) $\sum_{t\geq 0}\kappa(t)\Omega_2(t|z_i-z_j|)-\sum_{z_1,z_2\in I} a(I)=0$ for $z_1\neq z_2\in X$

(IEC) $\sum_{K\subset I} a(I)-\sum_{J\subset I} a(I)=0$ for every $K\cong J$.

Computer search: a witness graph on 24 vertices

- We performed a computer search starting from the Moser spindle: we build the graph incrementally, adding new vertices in every step.
- We employ beam search.
- We found a graph X on 24 vertices that testifies that $\delta \leq 0.247$ for any 1-avoiding set.
- Verification by symbolic calculation of the vertices and congruencies, and application of weak duality (basically solving the dual LP, augmented by error estimates).

The result of the beam search: X_{24}



Time to solve

This data is about the first n vertices of the set G_{24} in our paper.

n	# variables	# equations	time to solve (sec)	$\tilde{f}(0)$
7	12018	19	0.39	0.28305
8	12029	26	0.44	0.28258
9	12044	37	0.51	0.26631
$\vdots$				
21	19170	911	5.12	0.24997
22	22630	1288	6.68	0.24896
23	26899	2027	10.09	0.24796
24	34321	2375	13.46	0.24697

The beam search has run for a week on 128 CPUs.

The result of the beam search: $m_1(\mathbb{R}^2) \leq 0.247$

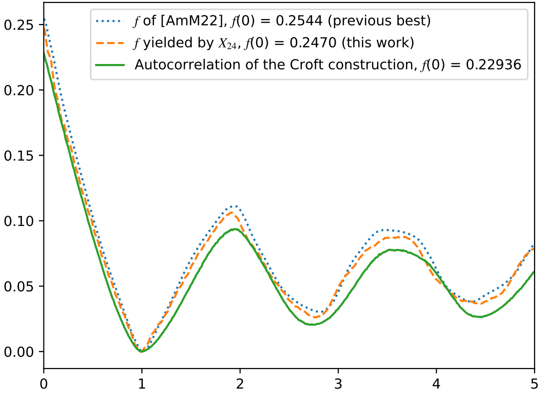
Theorem

The graph G_{24} is a witness to the fact that $m_1(\mathbb{R}^2) \leq 0.247$, settling Erdős's Conjecture.

The linear program defined by G_{24} has 22321 atom variables and 12000 Fourier variables. It has 24 (IE1) constrains, 227 (IE2) constraints connecting the Fourier variables to the atom variables, and 2122 (IEC) congruence constraints.

Plot of autocorrelation functions

If we plot the arising autocorrelation functions, we get is a nice indication of the progress on the upper bound on $m_1(\mathbf{R}^2)$.



Ongoing work

- We have now set our sights on other problems that can be attacked with this framework.
- Prove that at least 6 measurable colours are needed for the plane.
- Give density bounds on the sphere for sets avoiding orthogonal vectors ("double cap problem").

Summary

- Any finite point set on the plane can be turned into an upper bound on the density of a 1-avoiding set by solving some LP's.
- We found a graph on 27 vertices whose GFCN=4, which is interesting in itself. But could not find graphs with strictly larger values of GFCN.
- We invoke Fourier analysis, to put further constraints in the LP, and find a graph on 24 vertices which testifies that $\delta \leq 0.247$ for any 1-avoiding set
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References

- Gergely Ambrus, Adrián Csiszárik, Máté Matolcsi, Dániel Varga and Pál Zsámboki. The density of planar sets avoiding unit distances. Mathematical Programming 2023. https://link.springer.com/article/10.1007/s10107-023-02012-9

(Popularized in Quanta magazine more or less accurately...)

- Máté Matolcsi, Imre Z. Ruzsa, Dániel Varga, Pál Zsámboki. The fractional chromatic number of the plane is at least 4. Arxiv preprint 2023. https://arxiv.org/abs/2311.10069

Thank you!