Counting Unit Perimeter (& Area) Triangles

by Kenneth Moore

Based on a joint work with Ritesh Goenka and Ethan White

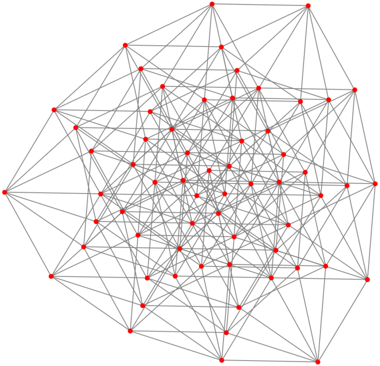
DGD^3 – 2024

Preface: repeated configuration problems

Fix a positive integer k and a property of k-tuples of points in $\mathbb{R}^2$. Often, it is an interesting problem to determine how many k-tuples in an n-point set in $\mathbb{R}^2$ can have the chosen property.

Preface: repeated configuration problems

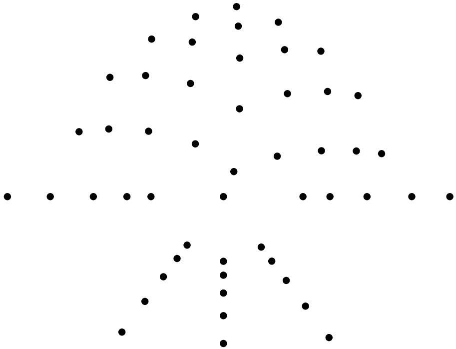
Fix a positive integer k and a property of k-tuples of points in $\mathbb{R}^2$. Often, it is an interesting problem to determine how many k-tuples in an n-point set in $\mathbb{R}^2$ can have the chosen property. One of the most famous instances of this form of problem is the unit distances problem, asked by Erdős in 1946. Here, the property is pairs of points being precisely distance one apart.



63 points, 246 unit distance pairs

Main question

Our main question: Given n points in the plane, how many unit perimeter or area triangles can they form?



A point set with many unit perimeter triangles

Notation

$u_a(n) = \max \#$ of unit area triangles formed by n points in $\mathbb{R}^2$

$u_p(n) = \max \#$ of unit perimeter triangles formed by n points in $\mathbb{R}^2$

- 'Triangle' is always non-degenerate.
- Symbols $\lesssim$ and $\approx$ mean $\leq$ and $=$ up to some absolute constant.

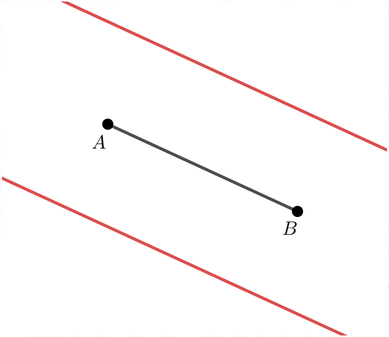
The lower bounds from grids

Erdős and Purdy (1971) showed that a section of the integer grid with dimensions $\frac{n}{\sqrt{\log n}} \times \sqrt{\log n}$ determines $\approx n^2 \log \log n$ triangles with the same area. Therefore, $u_a(n) \gtrsim n^2 \log \log n$

Pach and Sharir (2004) observed that a section of the section of the triangular lattice has $\approx n^{1+\frac{1}{\log \log n}}$ congruent equilateral triangles, which have the same perimeter by default. So, $u_p(n) \gtrsim n^{1+\frac{1}{\log \log n}}$.

Upper bounds

The current upper bounds in these problems come from incidence geometry. The idea is to start by fixing two points, and considering the set of third points that could possibly form a unit perimeter/area triangle.



For unit area triangles

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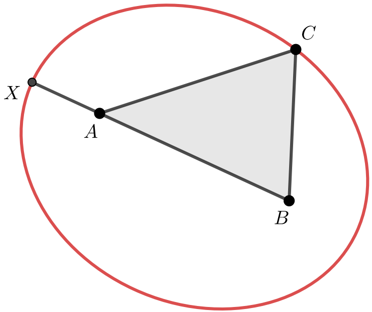

Ellipses are characterized by the property that $|AC| + |BC|$ is constant.

Upper bounds: Unit area

Theorem (Szemerédi-Trotter (1983))

Any collection of n points and m lines determine at most $\lesssim n^{2/3}m^{2/3} + n + m$ point-line incidences.

Idea: Define all $2\binom{n}{2}$ lines as described before. Every triangle present must result in at least 3 point-line incidences...

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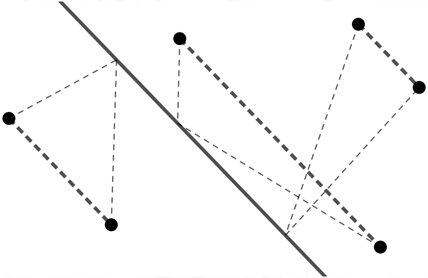
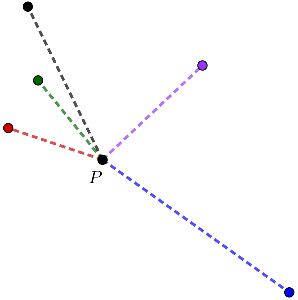
Idea: Define all $2\binom{n}{2}$ lines as described before. Every triangle present must result in at least 3 point-line incidences... It doesn't work! Lines need to be distinct to apply the theorem!



Upper bounds: Unit area

Pach and Sharir (1992) instead looked at each point P, and defined the $2(n-1)$ lines that come from segments including P. The resulting lines are all distinct (with one exception), and there are at most $\approx n^{4/3}$ incidences. Summing all points together gives $\approx n^{7/3}$ total incidences.



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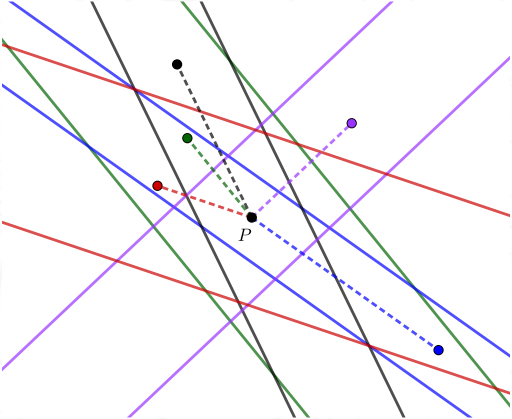

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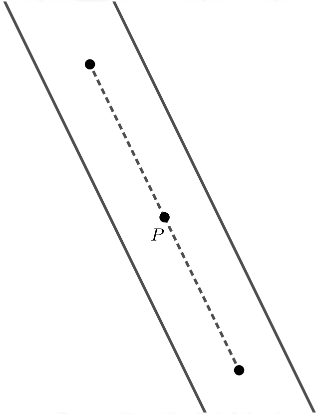

Upper bounds: Unit area

There are better upper bounds that we won't get to prove today; in fact, Raz and Sharir showed $u_a(n) \lesssim n^{20/9}$. So in summary, the maximum number of unit area triangles present in n points in the plane satisfies

$$n^2 \log\log n \lesssim u_a(n) \lesssim n^{20/9} \,.$$

Upper bounds: Unit perimeter

We can define all $\approx n^2$ ellipses that could form a unit perimeter triangle with each pair of points. Sharir and Zahl (2017) showed the following bound on the number of point-curve incidences, with some restrictions. Here k is the number of 'degrees of freedom' of the family, and d is the degree of the curves.

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$$|\mathcal{I}| \leq c_{k,d}\left(n^{\frac{2k}{5k-4}} m^{\frac{5k-6}{5k-4}+\varepsilon} + n^{2/3}m^{2/3} + n + m\right).$$

Normal ellipses have 5-degrees of freedom, but the extra condition fixing the major axis allows us to use $k = 4$. All of these ellipses are distinct this time, so the theorem gives at most $\approx n^{9/4+\varepsilon}$ incidences.

New unit perimeter stuff: Back to Grids

The great thing about a grid is that one occurrence of a triangle of fixed perimeter will give you a many translates of it. Therefore, it might be sensible to try and consider the maximum number of non-congruent triangles possible to fit in the grid.

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The great thing about a grid is that one occurrence of a triangle of fixed perimeter will give you a many translates of it. Therefore, it might be sensible to try and consider the maximum number of non-congruent triangles possible to fit in the grid.

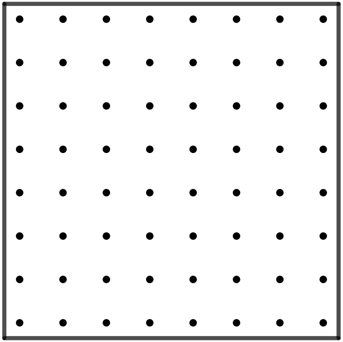
Lemma

For any $p > 0$, the number of non-congruent lattice triangles with perimeter p is at most $\lesssim p^{1+\frac{1}{\log\log p}}$.

Grid upper bound

Theorem

For $n \in \mathbb{N}$, the number of triangles with vertices in $[\sqrt{n}] \times [\sqrt{n}]$ having a fixed perimeter $p > 0$ is $\lesssim n^{\frac{3}{2}+\frac{1}{\log\log n}}$.



Note that $p < 4\sqrt{n}$, since all the triangles are contained in the square circumscribing the grid.

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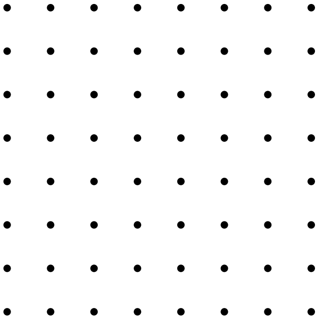

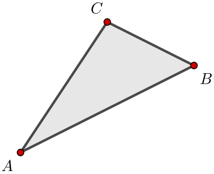

Around $\approx n^{\frac{1}{2}+\frac{1}{\log\log n}}$ congruence classes of triangles to choose.

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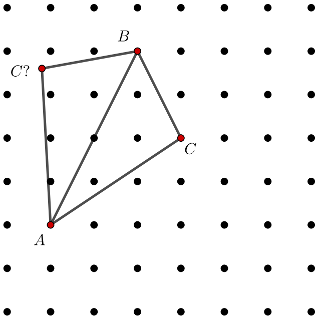

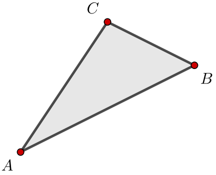

Grid lower bound

Now, it seems difficult to prove there really are a lot of triangles of the same perimeter in the grid, which is why Pach and Sharir only looked at a special case – equilateral triangles. We thought of a different type of special class of triangle to try: Heronian triangles. These are triangles with all integer side lengths, and integer area.

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Let $H(p)$ be the number of distinct Heronian triangles with perimeter p.

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Lemma

For $k \in \mathbb{N}$, we have $H(2(2k-1)^2) \gtrsim k$.

Grid lower bound

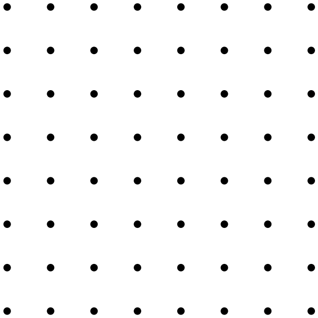
The reason we like Heronian triangles is that they work nicely with Heron's Area Formula:

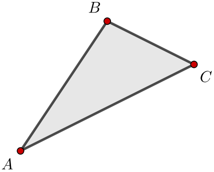
$$A = \sqrt{s(s-a)(s-b)(s-c)}\,.$$

Grid lower bound

Theorem

For $n \in \mathbb{N}$, there are $\gtrsim n^{\frac{5}{4}}$ triangles with vertices in $[\sqrt{n}] \times [\sqrt{n}]$ having the same perimeter.





Around $k \approx n^{1/4}$ Heronians to choose

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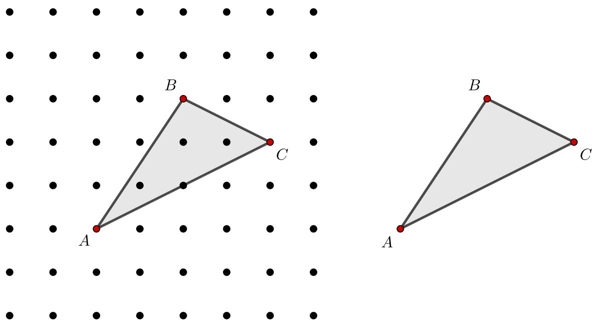

Around n choices of translate

Now unit perimeter stuff: Back to Grids

In summary, looking at sections of the grid, we have a lower bound of $\approx n^{\frac{5}{4}}$, and an upper bound of $\approx n^{\frac{3}{2}+\frac{1}{\log\log n}}$.

If you look at the data... it seems like there are $\approx n^{4/3}$ Heronian triangles?

A new lower bound

Let's start by identifying $\mathbb{R}^2$ with the complex numbers $\mathbb{C}$. Consider the map $f : \mathbb{C} \mapsto \mathbb{C}$, $z \mapsto \frac{z^2}{1+|z^2|}$.

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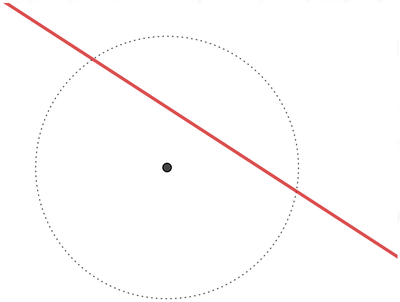

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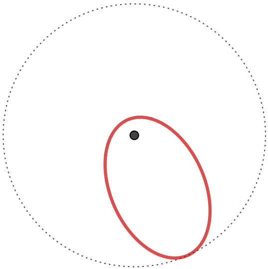

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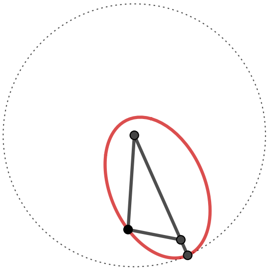

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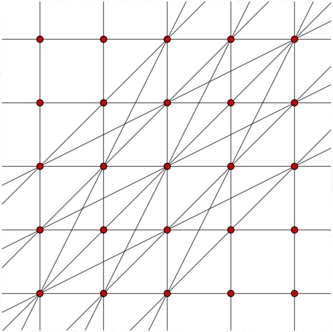

We know that the Szemerédi-Trotter theorem is tight: you can get a configuration of n points and n lines with roughly $n^{4/3}$ incidences.

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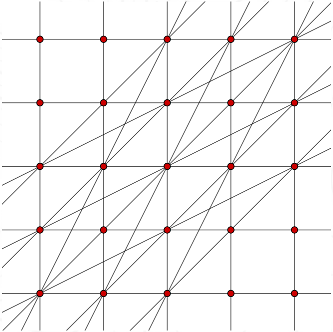
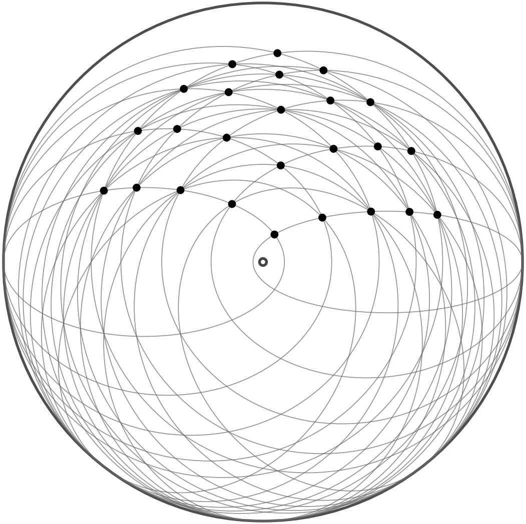


We know that the Szemerédi-Trotter theorem is tight: you can get a configuration of n points and n lines with roughly $n^{4/3}$ incidences.

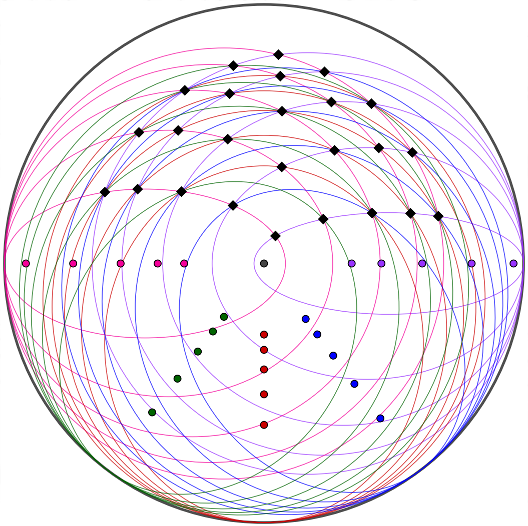
Let's start with an extremal configuration of n lines and n points, and apply the map $f(z) = \frac{z^2}{1+|z^2|}$ to the whole plane. This map sends all lines to ellipses, and the number of incidences is preserved.

A new lower bound



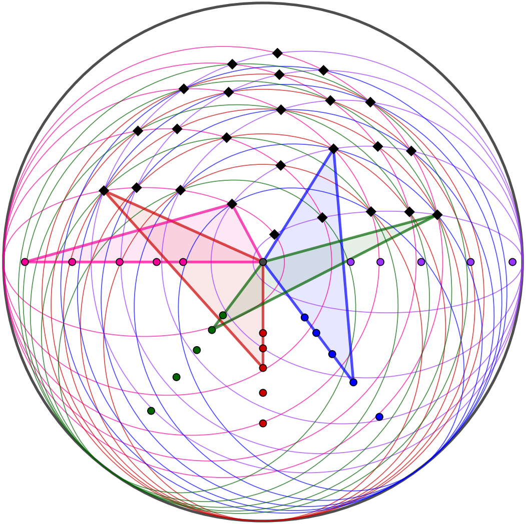
The resulting point set initially looks like this.

A new lower bound



We add the second foci of all the ellipses, and the origin.

A new lower bound



$2n + 1$ points total, and $\approx n^{4/3}$ unit perimeter triangles!

Conclusion

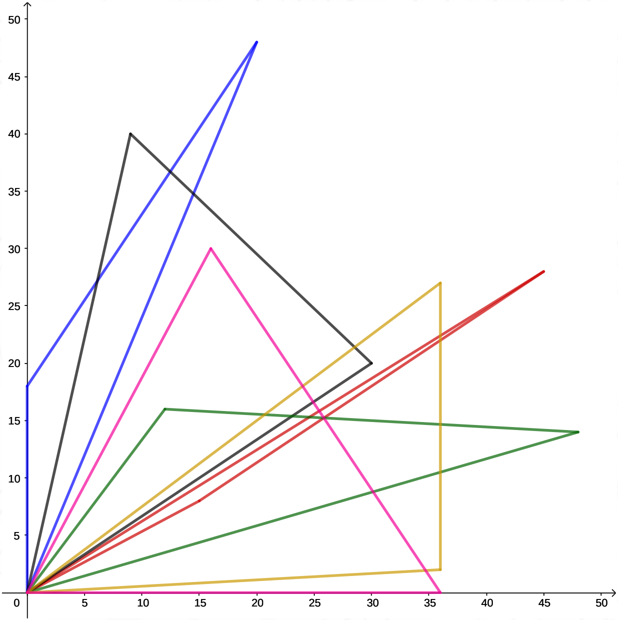
To conclude, the bounds for these problems stand at

$$n^{\frac{4}{3}} \lesssim u_p(n) \lesssim n^{\frac{9}{4}+\varepsilon},$$

$$n^{\frac{5}{4}} \lesssim u_{p,\mathrm{grid}}(n) \lesssim n^{\frac{3}{2}+\frac{1}{\log\log n}},$$

$$n^2 \log\log n \lesssim u_a(n) \lesssim n^{\frac{20}{9}}.$$

Thank you!



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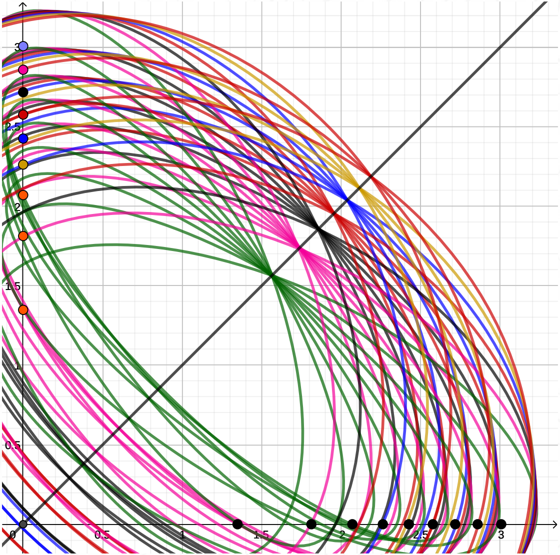
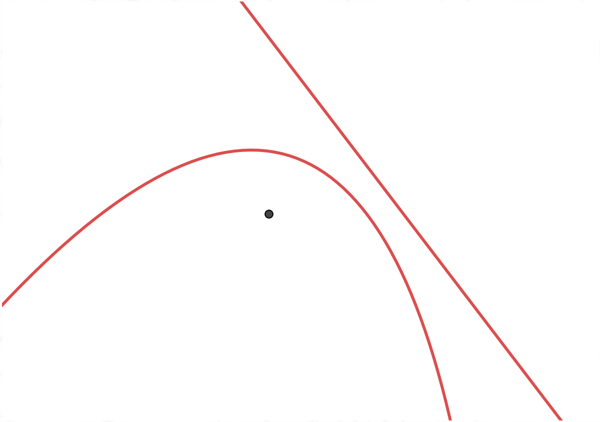


$3n$ points, $\frac{n(n-1)}{2}$ unit perimeter triangles??

A new lower bound

Let's start by identifying $\mathbb{R}^2$ with the complex numbers $\mathbb{C}$. Consider the map $f_1 : \mathbb{C} \mapsto \mathbb{C}$, $z \mapsto z^2$. If we start with a line in $\mathbb{R}^2$, this transformation sends it to a parabola, with its focus at the origin.



A line $\ell(t)$, and its image $\gamma_1(t)$ under f_1

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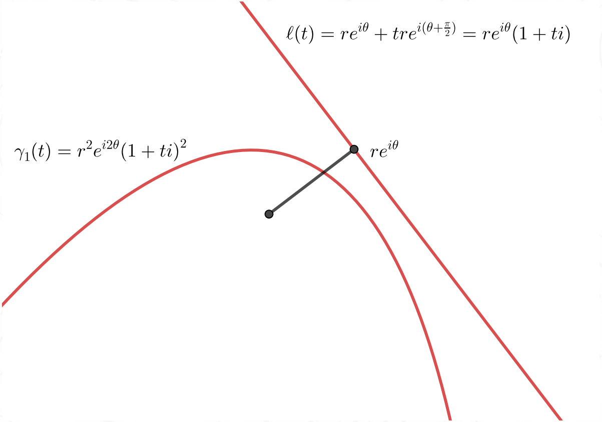

Write the equation for the supposed parabola with complex numbers

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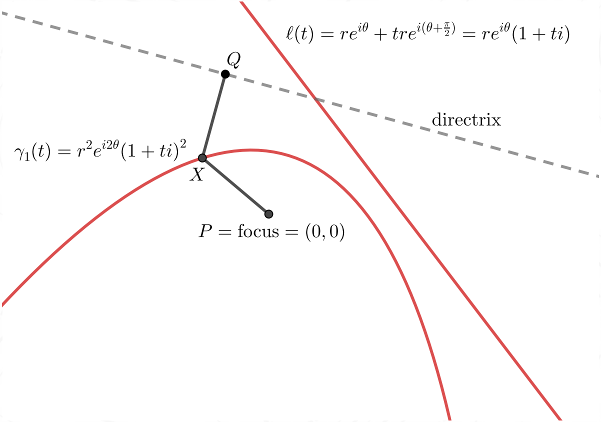
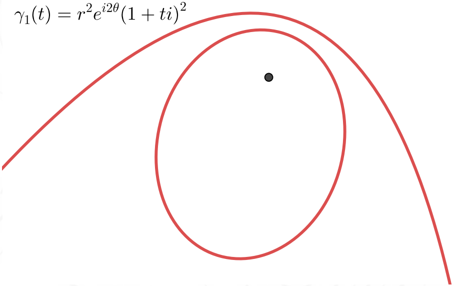


Parabolas are characterized by the property that $|XQ| = |XP|$.

A new lower bound

Next, we have our parabola. We now apply another map $f_2 : \mathbb{C} \mapsto \mathbb{C}$, $z \mapsto \frac{z}{1+|z|}$.



$\gamma_1(t)$, and its image $\gamma_2(t)$ under f_2

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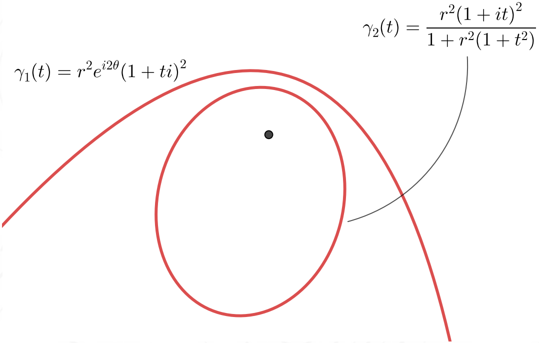

Write the equation for the supposed ellipse with complex numbers

A new lower bound

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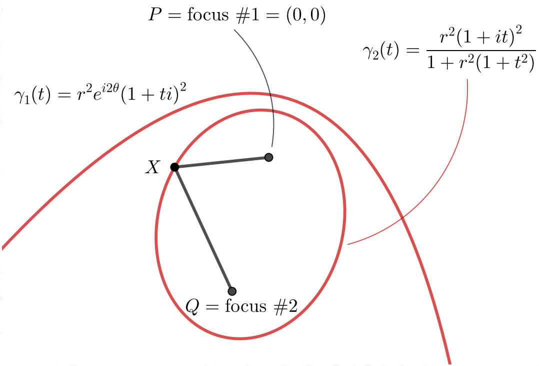

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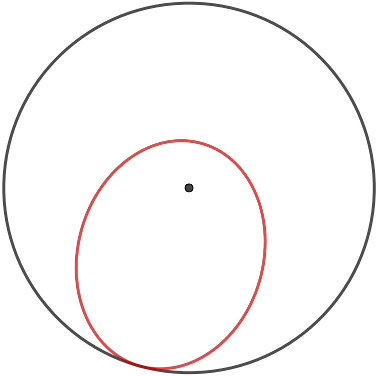

These guys are also tangent to the unit circle!

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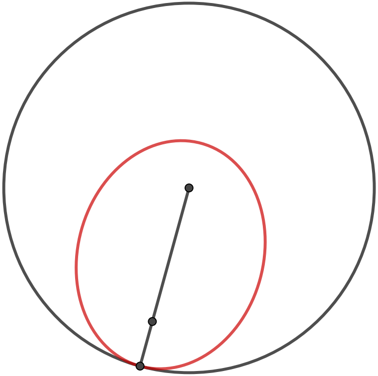

So this distance is 1...

A new lower bound

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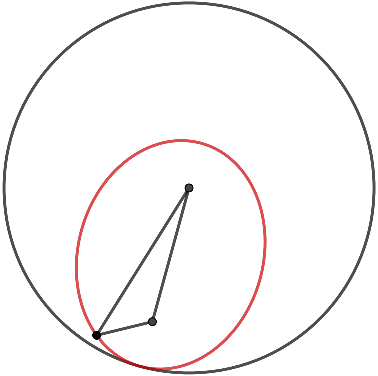

So any triangle formed by the two foci and a point on the boundary is perimeter 2.