Random spherical disc–polygons and a spherical spindle-convex duality

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Joint work with Viktor Vígh

Discrete Geometry Days^3, Budapest

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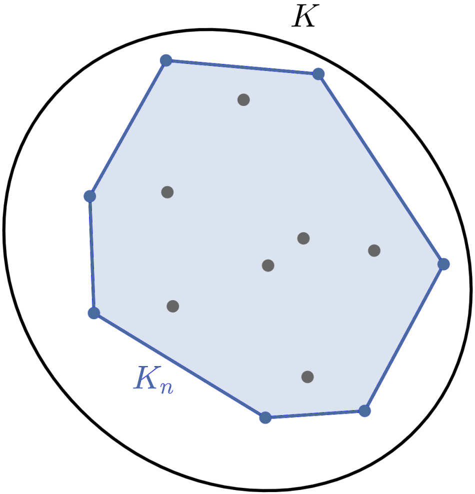
Random polytopes

- convex hull of n independently chosen uniform points from K



The notion of spindle-convexity

Mayer 1935; Polovinkin 1996; Bezdek, Lángi, Naszódi, Papez 2007

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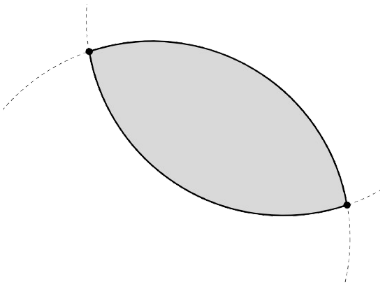
- *r spindle-convex set*: contains the r-spindle of every pair of its points



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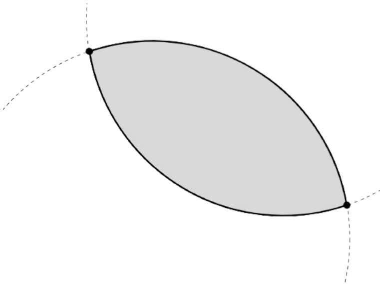
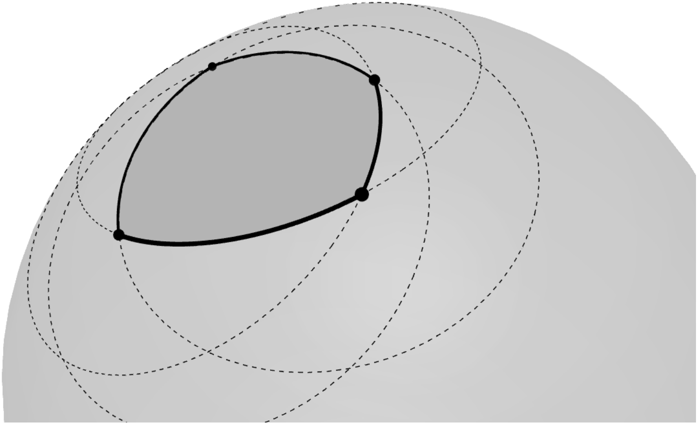
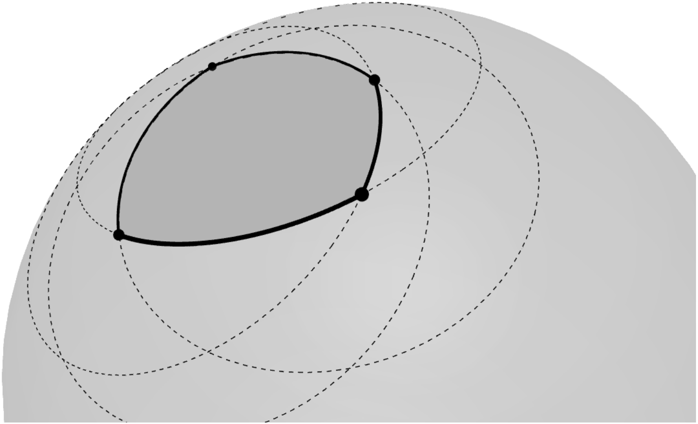
- *r-spindle* of two points: intersection of all closed discs of radius r that contain them
- *r spindle-convex set*: contains the r-spindle of every pair of its points
- *r disc-polygon*: spindle-convex hull of a finite set of points



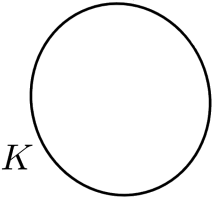
- Spherical spindle convexity → same definitions on the sphere S^2



- Spherical spindle convexity → same definitions on the sphere S^2
- Connection between Euclidean spindle-convexity and classical spherical convexity

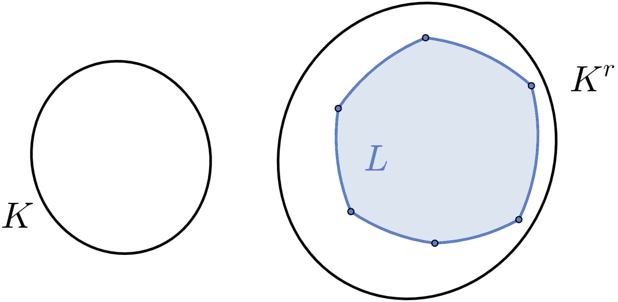


Constructing circumscribed disc-polygons



Constructing circumscribed disc-polygons

Let $K_{(n)}$ be the dual of a disc-polygon in K^r (the dual of K)



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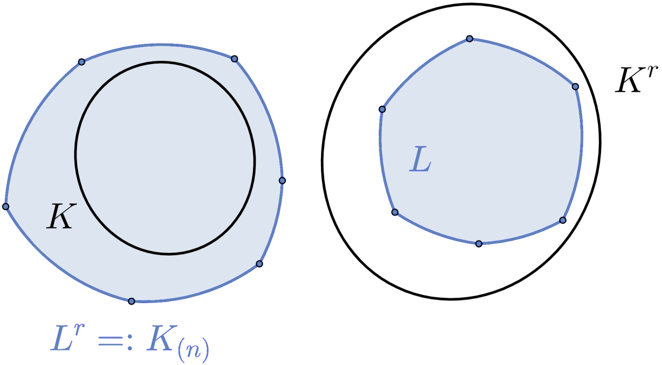

A spindle-convex duality - Euclidean case

For $K \subseteq \mathbb{R}^2$, let

$$K^r = \{y \in \mathbb{R}^2 \mid K \subseteq B(y,r)\} = \bigcap_{x \in K} B(x,r).$$

(Fodor, Kurusa, Vígh 2016,
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Analogous notion in L-convexity (N., Vígh 2023, Fodor, Grünfelder 2024)

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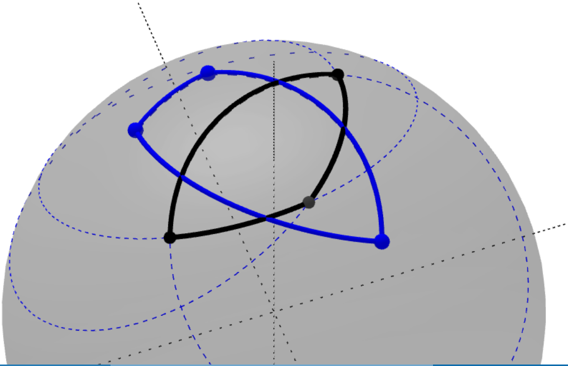

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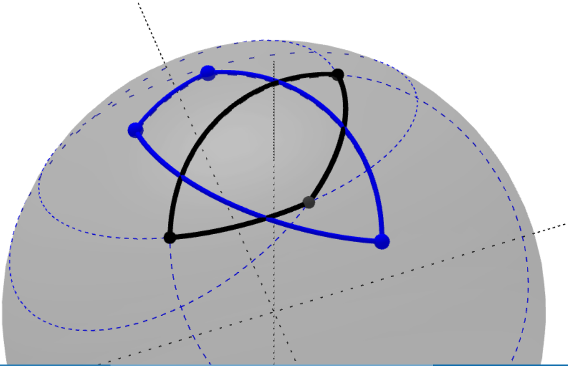
For $r = \pi/2$:

$$K^{\pi/2} = \{y \in S^2 \mid \langle x, y \rangle \geq 0 \; \forall x \in K\} = -K^\circ$$



Lemma (Fodor, Kurusa, Vígh 2016; Fodor, Vígh 2018)

Let $K \subseteq \mathbb{R}^2$ be an r spindle-convex disc. Then

$$(i)\ \operatorname{Per}(K^r) + \operatorname{Per}(K) = 2r\pi \quad \text{and}$$

$$(ii)\ \operatorname{Area}(K^r) = r^2\pi - r \cdot \operatorname{Per}(K) + \operatorname{Area}(K).$$

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Lemma (N., Vígh 2024+)

Let $K \subseteq S^2$ be a spherically r spindle-convex disc. Then

$$(i)\ \ \mathrm{Per}(K^r) = \sin r \cdot 2\pi - \cos r \cdot \mathrm{Per}(K) - \sin r \cdot \mathrm{SArea}(K) \quad \text{and}$$
$$(ii)\ \ \mathrm{SArea}(K^r) = (1 - \cos r) \cdot 2\pi - \sin r \cdot \mathrm{Per}(K) + \cos r \cdot \mathrm{SArea}(K).$$

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For the spherical polar $K^\circ = -K^{\pi/2}$ we have

$$\mathrm{Per}(K^\circ) + \mathrm{SArea}(K) = 2\pi.$$

Thank you for your attention!