Packing density of sets with only two non-mixed gaps

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joint with Arsenii Sagdeev

Discrete Geometry Days

2 July 2024

Outline

Tilings of $\mathbb{Z}$

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Definition

Let $S \subset \mathbb{Z}$ be a finite set. We say that S tiles $\mathbb{Z}$ if there exists $A \subset \mathbb{Z}$ such that for every $x \in \mathbb{Z}$ there are unique $a \in A$, $s \in S$ such that $x = a + s$.

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Set $S_1 = \{0, 2, 4\}$ tiles $\mathbb{Z}$, but $S_2 = \{0, 1, 3\}$ does not. So, not every set tiles $\mathbb{Z}$.

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- E. M. Coven, A. Meyerovitz (1999): Characterization of sets S that tile $\mathbb{Z}$, $|S| = p_1^{\alpha_1} p_2^{\alpha_2}$ with prime p_1, p_2.

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- I. Łaba , I. Londner (2022-2023): Characterization of tilings with a period of length $(p_1 p_2 p_3)^2$ with prime p_1, p_2, p_3.

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- A. Biró (2005): There is a period of length at most $\exp\{D^{1/3+\varepsilon}\}$, $D = \mathrm{diam}(S)$.
- J. P. Steinberger (2009): The period can grow faster than any power of the diameter of S.

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Density

Upper density $\overline{d}(A)$ and lower density $\underline{d}(A)$ of set $A \subset \mathbb{Z}$ are

$$\overline{d}(A) = \limsup_{n\to\infty} \frac{A[-n, n]}{2n+1}, \quad \underline{d} = \liminf_{n\to\infty} \frac{A[-n, n]}{2n+1}.$$

If $d(A) = \underline{d}(A) = \overline{d}(A)$, then A has density $d(A)$.

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- M. J. Golay (1972): For some sets $|S| = k$ we have
$$d_p(S) \leq \frac{2.646}{k^2}.$$

M-avoiding sets

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Integer distance graph

$G(\mathbb{Z}, M)$ is a graph with the vertex set $\mathbb{Z}$ where two vertices $v_1, v_2 \in \mathbb{Z}$ are adjacent if and only if $|v_1 - v_2| \in M$.

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Independence ratio of an integer distance graph

Independence ratio $\mu(M)$ of $G(\mathbb{Z}, M)$ is the maximum upper density of an M-avoiding set.

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Packing density and independence ratio

For finite $S \subset \mathbb{Z}$ and $M = \{s_2 - s_1 : s_1, s_2 \in S,\ s_1 < s_2\}$ we have $d_p(S) = \mu(M)$.

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Proof.

Two distinct translates $a_1 + S$ and $a_2 + S$ share a common point if and only if $a_1 + s_1 = a_2 + s_2$ for some $s_1 \neq s_2 \in S$, which can be rewritten as $a_1 - a_2 = s_2 - s_1$. □

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- S. Gupta (2000): for an arithmetical progression $M = \{n, n+d, n+2d, ..., n+(k-1)d\}$ we have

$$\mu(M) = \begin{cases} \frac{2n+(k-1)(d-1)}{2(2n+(k-1)d)}, & d \text{ is odd}, \\ \frac{1}{2}, & d \text{ is even}. \end{cases}$$

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- R. K. Pandey, A. Tripathi (2015): for a geometric progression $M = \{a^k, a^{k-1}b, ..., ab^{k-1}, b^k\}$ we have $\mu(M) = \mu(\{a, b\}) = \frac{\lfloor (a+b)/2 \rfloor}{2}$.

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- M. Axenovich, J. Goldwasser, B. Lidický et al. (2019): For any 4-elements set S we have $d_c(S) \leq \frac{1}{3}$. It is tight if $S = \{0, 1, 2, 4\}$.

Overview: Packings and coverings for $|S| = 3$

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Schmidt–Tuller conjecture for packings and coverings (2008)

Let $\lambda_1, \lambda_2 \in \mathbb{N}$ be two coprime integers. Then for $S = \{0, \lambda_1, \lambda_1 + \lambda_2\}$ we have

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The result for the packing density was proved in 2004 by D. D.-F. Liu and X. Zhu in terms of the maximum upper density of M-avoiding sets for $M = \{\lambda_1, \lambda_2, \lambda_1 + \lambda_2\}$.

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The result for the packing density was proved in 2004 by D. D.-F. Liu and X. Zhu in terms of the maximum upper density of M-avoiding sets for $M = \{\lambda_1, \lambda_2, \lambda_1 + \lambda_2\}$. In 2022, N. Frankl, A. Kupavskii, A. Sagdeev proved both results in a unified way.

Lower bound

Theorem 1

Let a, b be coprime positive integers. For $k, m \in \mathbb{N}$, let $d \in \mathbb{Z}$ and $0 \leq r \leq k + m$ be unique integers such that $a - b = (k + m + 1)d + r$. Then for $S = \{0, a, ..., ka, ka + b, ..., ka + mb\}$ and $M = \{ia + jb : 0 \leq i \leq k,\ 0 \leq j \leq m,\ i + j > 0\}$, we have

$$d_p(S) = \mu(M) \geq \begin{cases} \frac{1}{k+m+1}, & r = 0, \\ \frac{b+kd}{ka+(m+1)b}, & 1 \leq r \leq m, \\ \frac{a-m(d+1)}{(k+1)a+mb}, & m+1 \leq r \leq k+m. \end{cases}$$

Upper bound

Theorem 2

Let a, b be coprime positive integers. For $k \in \mathbb{N}$, let $d \in \mathbb{Z}$ and $0 \le r \le k+1$ be unique integers such that $a - b = (k+2)d + r$. Then for $S = \{0, a, \dots, ka, ka+b\}$ and $M = \{a, \dots, ka\} \cup \{b, a+b, \dots, ka+b\}$, we have

$$d_p(S) = \mu(M) = \begin{cases} \frac{1}{k+2}, & r = 0, \\ \frac{b+kd}{ka+2b}, & r = 1, \\ \frac{a-d-1}{(k+1)a+b}, & 2 \le r \le k+1. \end{cases}$$

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Set $A = \{t(a-b) : 0 \le t < b + kd\} + n\mathbb{Z}$, $n = ka + (m+1)b$, is S-packing of the required density.

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Lemma (Haralambis, 1977)

Let $M \subset \mathbb{N}$ and $\alpha \in (0,1)$ be a real number. Suppose that for every M-avoiding set M with $0 \in A$ there exists n such that $A[n] = |A \cap [0, n-1]| \le n\alpha$. Then $\mu(M) \le \alpha$.

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On the upper bound

We prove that at least one of the integers $n_1 = ka + (m+1)b$ and $n_2 = (k+1)a + mb$ is suitable for every M-avoiding set A with $0 \in A$.

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Conjecture (D. D.-F. Liu and X. Zhu, 2004)

If $M = \{x, y, y - x, x + y\}$, where $x < y$, $x = 2k + 1$, $y = 2m + 1$, then $\mu(M) = \frac{(k+1)m}{4(k+1)m+1}$.

Remark: for x, y of distinct parity, we have $\mu(M) = 1/4$.

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Sharp asymptotic bounds on the packing density $d_p(S)$ of k-element subsets $S \subset \mathbb{Z}$ where $k \to \infty$.

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Problem 4

Tight upper bounds on $d_c(S)$ for k-element subsets $S \subset \mathbb{Z}$, $k \geq 5$? Tight lower bounds on $d_p(S)$ for k-element subsets $S \subset \mathbb{Z}$, $k \geq 4$?

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Conjecture (B. Bollobás, S. Janson, O. Riordan, 2010)

For any 5-element set S we have $d_c(S) \leq \frac{3}{11}$. For any 6-element set S we have $d_c(S) \leq \frac{1}{4}$.

Open problems

Problem 5

Can we find an explicit expression for the maximum upper density $\mu(M)$ of an M-avoiding set for an arbitrary triple $M = \{a, b, c\} \subset \mathbb{N}$?