News on polychromatic colorings

Dömötör Pálvölgyi

Hogwarts School of Witchcraft and Wizardry

Discrete Geometry Days^3

BME 2024

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Before abstract definitions, geometric examples.

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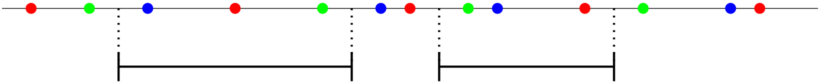

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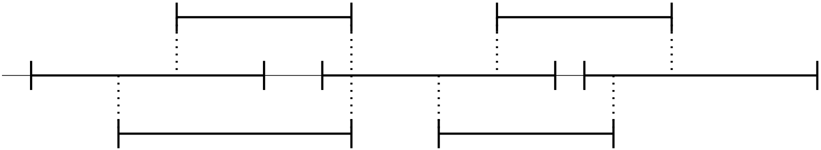

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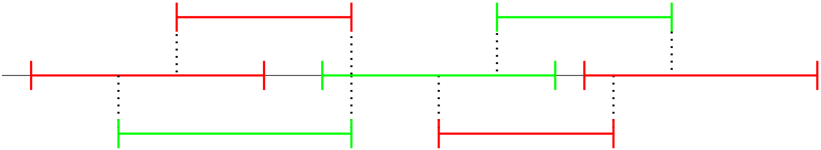

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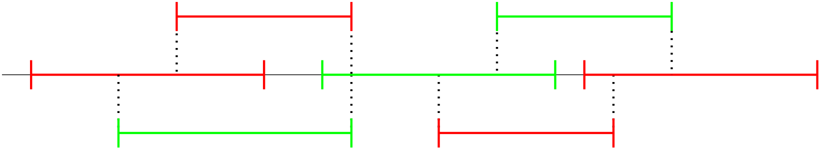

Lower are called *dual* range spaces—in this talk we mainly consider upper, so-called *primal* range spaces, i.e., coloring points.

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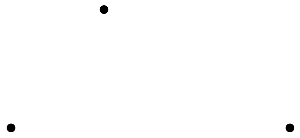

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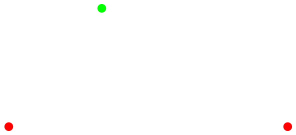

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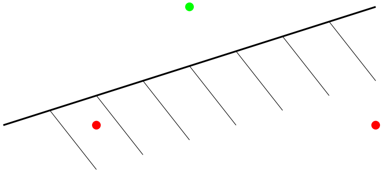

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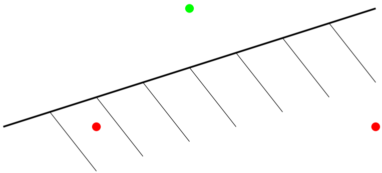

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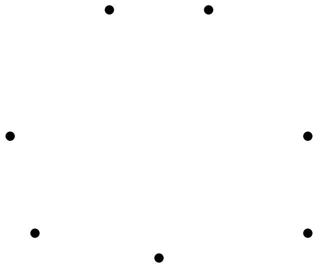

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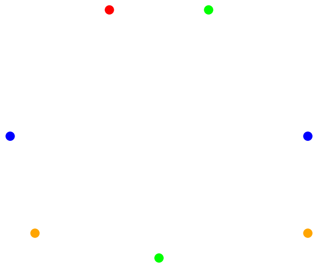

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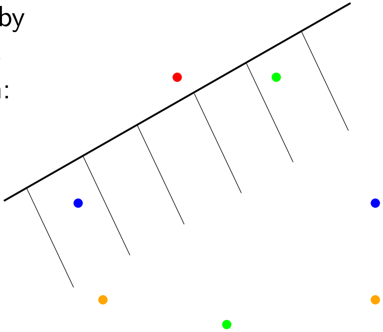

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For every k, m there is a finite $X \subset \mathbb{R}^2$ such that for every k-coloring of X there is a line that contains at least m points from X and all have the same color.

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From the point-line duality of $\mathbb{R}^2$ we get:

Theorem (Pach-Tardos-Tóth '05)

For every k, m^* there is a finite collection of lines $\mathcal{L}$ in the plane such that for every k-coloring of $\mathcal{L}$ there is a point contained in at least m^* lines from $\mathcal{L}$ and all have the same color.

Abstract Definitions

For a hypergraph $\mathcal{H} = (V, \mathcal{E})$, denote by m_k the smallest number for which we can k-color any finite $X \subset V$ such that for any $E \in \mathcal{E}$ with $|E \cap X| \geq m_k$ all k colors occur in $E \cap X$.

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From defs: $\chi_{fat} = 2 \iff m_2 < \infty$

$m_k \leq m_{k+1}$

Back to Geometry: Translates

<u>Decomposition of multiple packing and covering</u>

by

János Pach

In this talk I would like to call attention to a branch of problems which seem to appear collaterally in combinatorics and discrete geometry.

Let X be an underlying set and let <u>S</u> be a family of subsets of X. Further, let k be a cardinal. The set system <u>S</u> is said to form a k-<u>fold packing</u> if each

The simplest unsolved question is the following.

PROBLEM (vi). Is it true that, for a sufficiently large k, every k-fold covering of the plane with equal circles is decomposable into two coverings?

Conjecture (Pach '80)

Thick coverings of the plane by translates of any convex planar range are decomposable into two coverings.

Translates in $\mathbb{R}^2$

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For every planar convex set D there is an m such that we can color any finite $X \subset \mathbb{R}^2$ with two colors such that every translate of D with at least m points contains both colors, i.e., $\chi_{fat} = 2$.

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Theorem (P. '13, Pach-P. '16)

For every m there is a finite $X \subset \mathbb{R}^2$ such that for every two-coloring of X there is a unit disk that contains at least m points from X and all have the same color, i.e., $\chi_{fat} > 2$.

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Is Pach's conjecture true if we can use more colors?

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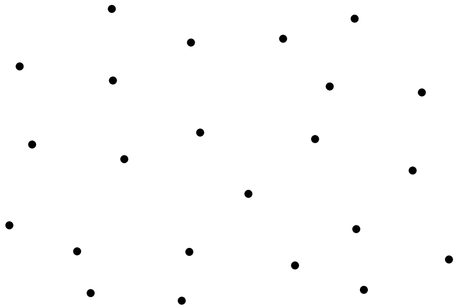

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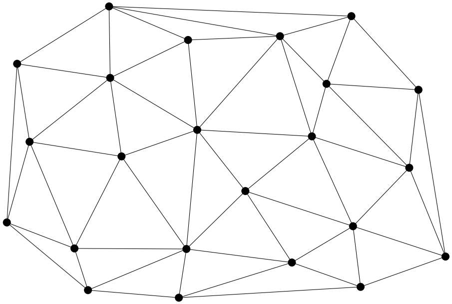

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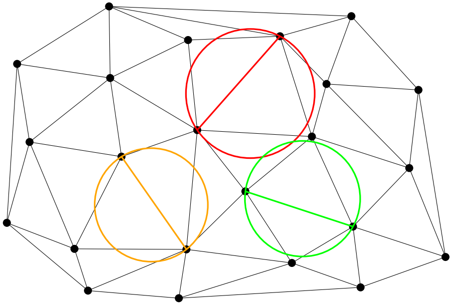

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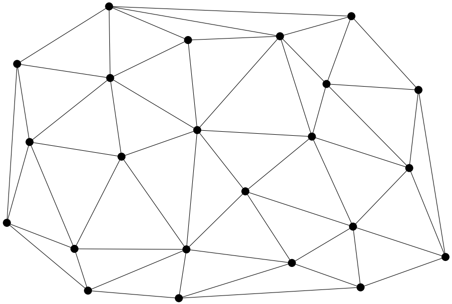

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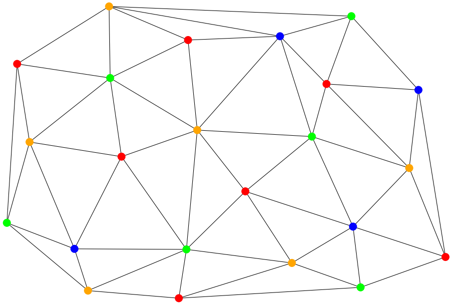

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We can apply the Four Color Theorem. □

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Could even be true with 2 colors for homothets of convex polygons;

known for homothets of triangles (Keszegh-P. '11),

known for homothets of squares (Ackerman-Keszegh-Vizer '16).

What about disks?

Constructions for Disks

Theorem (Pach-Tardos-Tóth '05)

For every m there is a finite $X \subset \mathbb{R}^2$ *such that for every 2-coloring of* X *there is a disk that contains at least m points from* X *and all have the same color, i.e.,* $\chi_{fat} > 2$.

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For translates of any convex planar shape $\chi_{fat} \leq 3$*.*

Summary for Disks

	unit disks	**any disks**
stabbed	$\chi_{fat} = 2$ $m_k = O(k)$ Damásdi-P. '22	$\chi_{fat} = 3$ Damásdi-P. '22 Ackerman-Keszegh-P. '19
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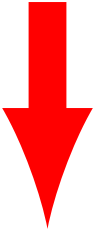
THIS GUY IS AN IDIOT



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Three Dimensions

Theorem (P. '10)

For every polyhedron $P \subset \mathbb{R}^3$ *and* m *there is a finite* $X \subset \mathbb{R}^3$ *such that for every two-coloring of* X *there is a monochromatic translate of* P *that contains at least* m *points from* X*, i.e.,* $\chi_{fat} > 2$.

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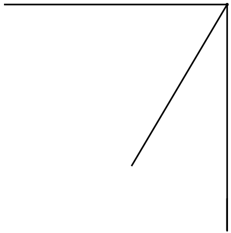

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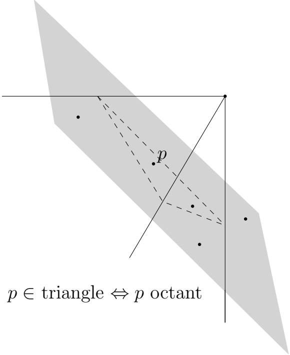

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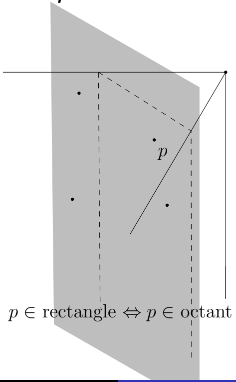

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Follows from realizing all axis-parallel rectangles and from

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Bottomless Rectangles

Theorem (Keszegh '11, Asinowski, Cardinal, Cohen, Collette, Hackl, Hoffmann, Knauer, Langerman, Lason, Micek, Rote, Ueckerdt '13)

Any finite $X \subset \mathbb{R}^2$ can be k-colored such that any axis-parallel bottomless rectangle with $m_k \leq 3k - 2$ points contains all k colors.

Theorem (Cardinal, Knauer, Micek, Ueckerdt (+ KP) '15)

Any finite collection of axis-parallel bottomless rectangles can be k-colored such any point covered by $m_k^* = O(k^{5.09})$ rectangles is covered by all k colors.

We do not know matching lower bounds.

Theorem (Cardinal, Knauer, Micek, P., Ueckerdt, Varadarajan)

$m_k^* \leq 2k - 1$ in many special cases.

Hereditary k-colorability of Abstract Hypergraphs

For a hypergraph $\mathcal{H} = (V, \mathcal{E})$, denote by m_k the smallest number for which we can k-color any finite $X \subset V$ such that for any $E \in \mathcal{E}$ with $|E \cap X| \geq m_k$ all k colors occur in $E \cap X$.
For intervals $m_k = k$, for halfplanes $m_k = 2k - 1$.
$m_k = \infty$ is possible, e.g., for lines.
Important: In the induced hypergraph $\mathcal{H}|_X$ we only care about m_k-heavy edges. This is m_k-fat induced subhypergraph of $\mathcal{H}$.

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Can we bound m_k with a function of m_2?

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SEE BLACKBOARD!

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For dual bottomless rectangles and dual halfplanes:
$m_2^* = 3$ but we don't know whether $m_k^* = 2k - 1$.

New Abstract Construction

Theorem (P. '23+)

There exists a 5-uniform hypergraph that has no polychromatic 3-coloring, but its 3-fat induced subhypergraphs are 2-colorable. Therefore, $m_3 = 6$ and $m_2 = 3$.

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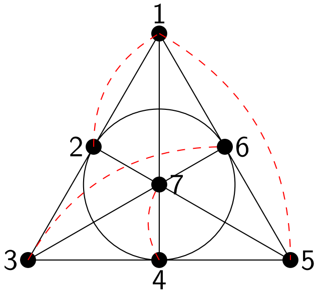

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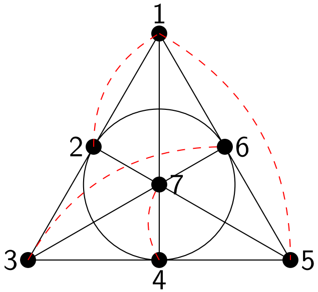

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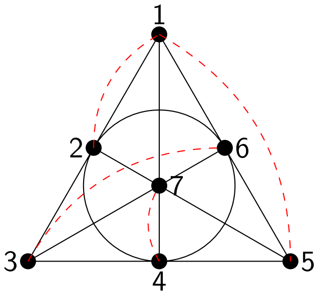

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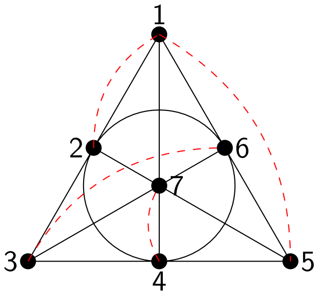

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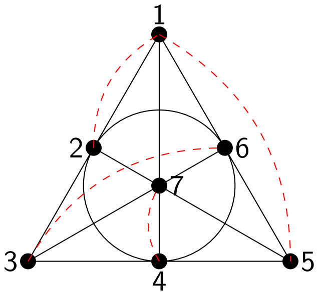

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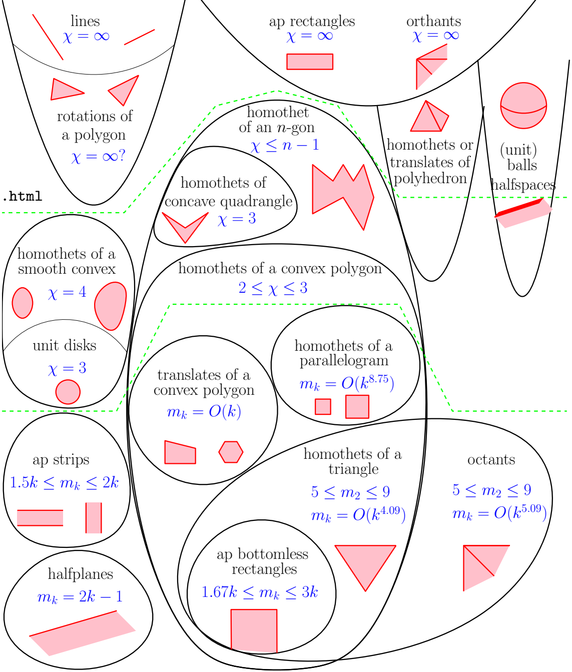
Does $m_2 < \infty$ imply $m_k < \infty$?

Summary of known results

for all results, see
`coge.elte.hu/cogezoo.html`



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