Generalisations of the Erdős-Szekeres Theorem

Attila Pór

Discrete Geometry Days[3]

July 5, 2024

Outline

- Erdős-Szekeres Theorem
- Ramsey-type questions
- Definition of hypersequences and orientation/coloring
- Justification via application
- some weak results

Erdős-Szekeres Theorem

Theorem

Given a sequences of $n^2 + 1$ distinct numbers, there is a monotone subsequence of length $n + 1$.

Ramsey type results

Theorem

For every n, k, m there exists $N = N(n, k, m)$ such that if we color each k-tuple of a sequence of length N with at most m colors, there exists a monochromatic subsequence of length n.

The set of colors is M, where $m = |M|$ and the coloring is
$c : \binom{N}{k} \to M$
In general we are interested for which $P \in M$ are there arbitrary long sequences, where each k-tuple has color P?

Hyper sequence

Definition

A hyper sequence of length n is a function on all r element subsets of $[n]$.

For example, if $n = 4, r = 2$ we have six elements:
$\{a_{1,2}, a_{1,3}, a_{1,4}, a_{2,3}, a_{2,4}, a_{3,4}\}$.
A subsequence is a restriction of the function to a subset of $[n]$.

Orientation in $\mathbb{R}^d$

Definition

The orientation of any $d+1$ points in $\mathbb{R}^d$ in general position is either positive or negative.

Erdős-Szekeres in higher dimension

Given a sequence of points in $\mathbb{R}^d$ in general position, color the $d+1$ tuples according to their orientation.
A monochromatic sequence is the vertex set of a cyclic polytope.
One can also say that those points are along some convex curve.

coloring Hypersequences of points in $\mathbb{R}^d$

We want to record the orientation of any $d+1$ points.
Since each point is related to an r element set, therefore any $d+1$ tuple of points is related to a set of at MOST $(r \cdot (d+1))$ elements.
Observe that the r element sets may not be disjoint.
There is a finite set M of colors related to every possible orientation of all the $d+1$-tuples one can form from a $(r \cdot (d+1))$ long hypersequence.

Definition

Let $a : \binom{[n]}{r} \to \mathbb{R}^d$ be a hypersequence. The orientation of every $d+1$ tuple is captured by the coloring
$c : \binom{[n]}{r \cdot (d+1)} \to M$

Use Ramsey....

Example $d = 1, r = 2$

Let a be a hypersequence of numbers where $r = 2$.
Assume that a is monochromatic. Point pairs with the same order:
$a(1,2), a(3,4)$ and $a(10,21), a(22,40)$
$a(1,3), a(2,4)$ and $a(8,21), a(19,40)$
$a(1,4), a(2,3)$ and $a(10,21), a(13,17)$
$a(1,2), a(1,3)$ and $a(10,21), a(10,40)$
$a(1,2), a(2,3)$ and $a(10,21), a(21,33)$
$a(1,3), a(2,3)$ and $a(10,21), a(13,21)$
That is ALL possible pairs.
For each there are two possible orientations, so at MOST 2^6 different colors.
But not all of them are possible, and yield arbitrary long sequences.
In fact, there are only 8.

Characterization in 1-dimension

Theorem

If $d = 1$, there are $2^r \cdot r!$ different monochromatic colorings that appear in arbitrary long monochromatic sequences.

Pick an ordering (of imporatance) of $[r]$ and pick a sign $\{+, -\}$ for each element.
To compare two r-element subsets, compare their "most" important element (like smallest element or second smallest element).
If those elements are different then order them the same way if that element's sign is positive and the opposite way if it is negative.
If those elements are the same, go to the next important element, ...

Line transversal problem for pairwise intersecting family of Convex sets in $\mathbb{R}^3$

Question: Given a Family of pairwise intersecting convex sets in $\mathbb{R}^3$ can we find a "large" subset that has a common line transversal?

Statement

STATEMENT(k): There exists $n = n(k)$ such that each family of pairwise intersecting convex bodies of size n contains a subfamily of size k that has a line transversal.

result?

Theorem

If STATEMENT(13) is true, than for every k STATEMENT(k) is true.

Connection

Given a family of pairwise intersecting convex sets $F_1, \dots, F_N$ in $\mathbb{R}^3$, pick a point $a(i,j) \in F_i \cap F_j$ to attain a hypersequence with $r = 2$ in $\mathbb{R}^3$.
Pick a monochromatic subsequence and "assume" that
- each set F_i is the convex hull of its points $(1,i),(2,i),\dots,(i,n)$
- each two sets intersect in only one point $a(i,j) = F_i \cap F_j$.

A "good" example where further investigation would not be neccessary:
Assume that the line through $a(1,2)$ and $a(10,11)$ intersects the triangle $a(3,5), a(4,5), a(5,18)$.
Than there is a line transversal to "almost" every set (in this case $n-3$).

Why 13?

If a family of convex sets has a common line transversal, then there is one in extreme position:

- through two vertices
- through a vertex and two edges
- through four edges

Don't worry, just Ramsey.

The case of four edges

Each edge is defined by three sets
Four edges is twelve sets
If it intersects one more set (other then the first twelve)
than it intersects almost all.

The end

Thank You