Variations of the Menger Embedding Theorem

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joint work with
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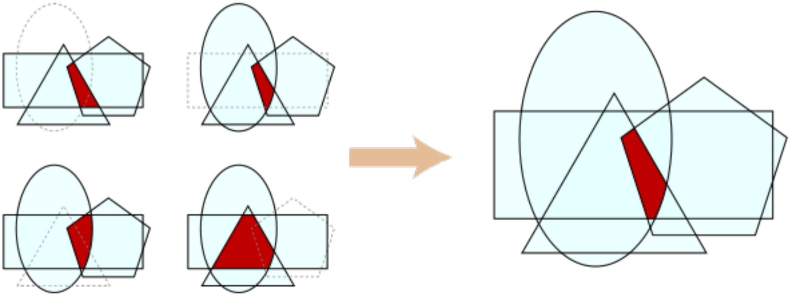
Two classical results

Helly's theorem (1921)

Let $\mathcal{F}$ be a collection of n convex bodies in $\mathbb{R}^d$.

$$\begin{matrix}\text{Every } d+1 \text{ of them} \\ \text{share a common point}\end{matrix} \Rightarrow \begin{matrix}\text{all of them} \\ \text{share a common point.}\end{matrix}$$



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Menger's embedding theorem (1928)

Let $\mathcal{M}$ be a metric space on $n \geq d+4$ points.

Every $(d+2)$-element subset is embeddable into $\mathbb{R}^d$ $\Rightarrow$ the whole $\mathcal{M}$ is embeddable into $\mathbb{R}^d$.

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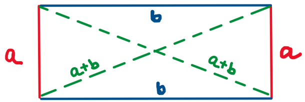
Both $d+2$ and $d+4$ are tight.



Fractional variant

Fractional Helly theorem (Katchalski, Liu '1979)

Let $\mathcal{F}$ be a collection of n convex bodies in $\mathbb{R}^d$.

α-fraction of $(d+1)$-tuples share a common point	$\Rightarrow$	some βn bodies share a common point.

Moreover, $\beta = \beta_d(\alpha) = 1 - (1-\alpha)^{1/(d+1)}$ (Kalai '1984).

The ℓ_∞-distance between (x_1, y_1) and (x_2, y_2) is defined by $\max(|x_1 - x_2|, |y_1 - y_2|)$.

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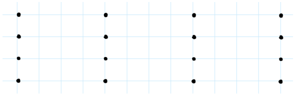
Fractional Menger theorem?

No: $\{m, 2m, \ldots, m^2\} \times \{1, 2, \ldots, m\} \subset \mathbb{R}^2$ equipped with the ℓ_∞-metric*.

- Almost all triples are embeddable into $\mathbb{R}$,
- No $(m+1)$-subset is embeddable into $\mathbb{R}$.



*The ℓ_∞-distance between (x_1, y_1) and (x_2, y_2) is defined by $\max(|x_1 - x_2|, |y_1 - y_2|)$.

(p, q)-variant

Helly (p, q)-theorem (Alon, Kleitman '1992)

Let $\mathcal{F}$ be a collection of n convex bodies in $\mathbb{R}^d$, $p \geq q \geq d + 1$.

among every p of the bodies some q share a common point $\Rightarrow$ some $c(p, q)$ points pierce $\mathcal{F}$.

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Menger (p, q)-theorem?

For which $p \geq q \geq 3$, for every metric space $\mathcal{M}$ on n points

among every p points some q are embeddable into $\mathbb{R}$ $\Rightarrow$ $\mathcal{M}$ is partitioned into $c(p, q)$ embeddable into $\mathbb{R}$ subsets ?

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Negative direction

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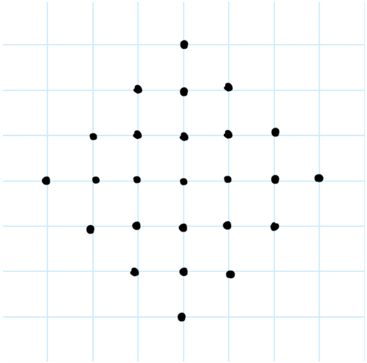

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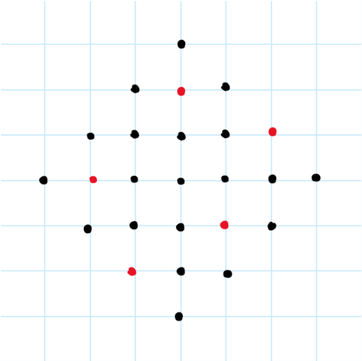

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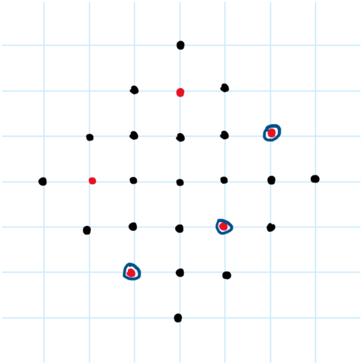

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Positive direction

Menger (p, q)-theorem holds for:

- $(3k, 2k + 1)$

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Menger (p, q)-theorem holds for:

- $(3k, 2k + 1)$ with 1 line and $k - 1$ singletons;

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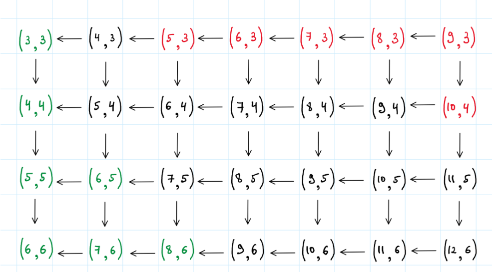
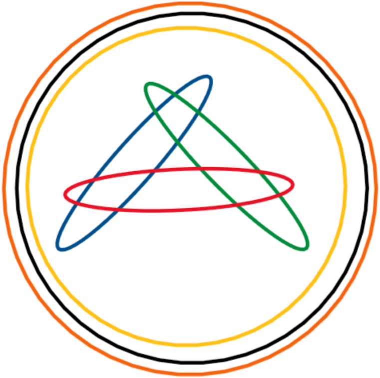


Variation: testing with fewer queries

Helly: at least one $(d + 1)$-tuple has not been checked
$\Rightarrow$ there might be no common point.



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Menger: $\mathcal{S} \subset \binom{[n]}{d+2}$ *tests the embeddability* if for every metric on $[n]$,

$$\begin{matrix}\text{every } S \in \mathcal{S} \\ \text{is embeddable into } \mathbb{R}^d\end{matrix} \quad \Rightarrow \quad \begin{matrix}\text{the whole space} \\ \text{is embeddable into } \mathbb{R}^d.\end{matrix}$$

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Theorem (Chvátal, de Rancourt, Gamboa Quintero, Kantor, Szabó)

$$\binom{n}{2} \leq f_1(n) \leq \left(2 + \frac{1}{3}\right)\binom{n}{2}.$$

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Theorem

$$\frac{3}{d+2}\binom{n}{d+1} \leq f_d(n) \leq \left(2 + \frac{1}{d+2}\right)\binom{n}{d+1}.$$

Open Problem

Menger's embedding theorem (1928)

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- $M(\mathbb{R}^2, \ell_\infty) \leq 6$ (Bandelt, Chepoi '1996)
- $M(\mathbb{R}^3, \ell_\infty) = \infty$ (Edmonds '2008)

Variations of the Menger Embedding Theorem

Helly's theorem

- For all $\alpha < 1$, fractional theorem holds and $\beta_d(\alpha) = 1 - (1-\alpha)^{1/(d+1)}$
- For all $p \geq q \geq 3$, (p,q)-theorem holds.
- Testing with fewer queries is not possible.

Menger's embedding theorem

- For all $\alpha < 1$, fractional theorem does not hold.
- If $p < \frac{8}{5}q$, then (p,q)-theorem holds; If $p > q^2$, then (p,q)-theorem does not hold.
- $\binom{n}{2} \leq f_1(n) \leq \left(2 + \frac{1}{3}\right)\binom{n}{2}$

Thank you for your attention!