Some inequalities on reduced convex bodies in the hyperbolic space

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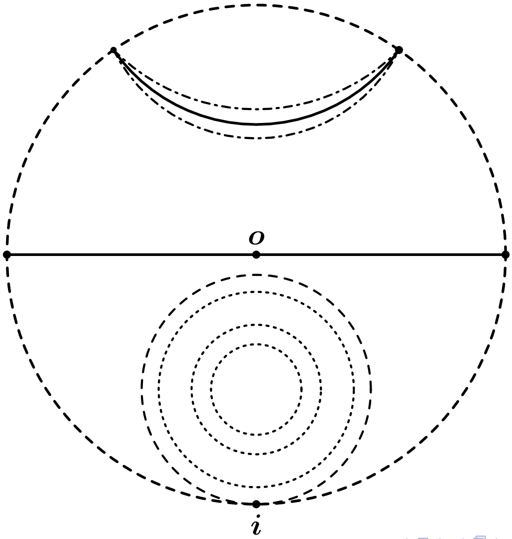
The Poincaré disk



Width in the hyperbolic space

- Let $K \subset \mathbb{H}^n$ a convex body and H a supporting hyperplane
- The hyperplane H' is *ultraparallel* to H if they are parallel with no common ideal point
- H and H' have a unique line which is orthogonal to both
- The *width* of K with respect to H is

$$w(K, H) = d(H, H')$$

  where H' is an ultraparallel hyperplane at the largest distance to H
- For any supporting hyperplane H, $w(K, H)$ equals the distance of H and the parallel supporting hypersphere to H

Reduced convex bodies

- $w(K)$ denotes the minimal width of K
- K is *reduced* if for all $L \subsetneq K$, $w(L) < w(K)$
- $L \subseteq K$ is a *reduction* of K if it is reduced of the same width
- Every convex body has a reduction
- In $\mathbb{R}^n$, the only centrally symmetric reduced bodies are balls and there is no known reduced polytopes in $\mathbb{R}^n$ for $n \geq 4$
- In $\mathbb{R}^2$ and in $\mathbb{S}^2$ polygons are reduced if and only if for every vertex v_i their projections t_i are interior points of the opposite side and $d(v_i, t_i) = w$

Pál's inequality

Theorem (Pál 1921, Bezdek–Blekherman 1999)

Among convex bodies of thickness $w > 0$ in $\mathbb{R}^2$ and in $\mathbb{S}^2$ if $w \leq \frac{\pi}{2}$, the regular triangle of height w is the unique body whose area is the smallest.

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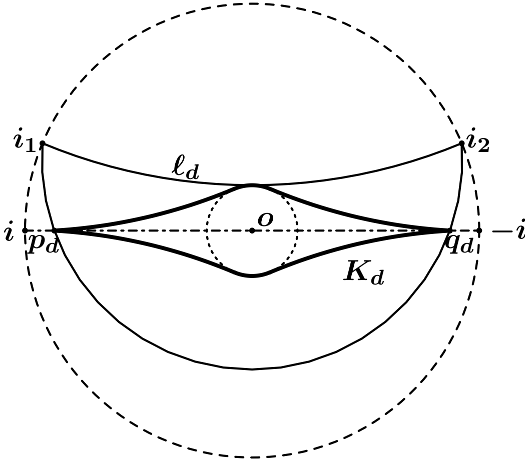
Theorem (Böröczky–Freyer–S. 2024+)

Let $w > 0$ be a fixed positive number. Then,

$$\inf \{\mathrm{vol}_n(K) : K \subset \mathbb{H}^n \text{ convex body}, w(K) \geq w\} = 0.$$

There is no solution in $\mathbb{H}^n$



Blaschke–Lebesgue–Leichtweiss Theorem

Theorem (Lebesgue 1914, Blaschke 1915)

Among convex bodies of constant width $w > 0$ in $\mathbb{R}^2$, the Reuleaux triangle of diameter w is the unique one with minimal area.

Theorem (Araújo 1997, Leichtweiss 2005, Bezdek 2021, Böröczky–S. 2022)

For any convex body K in $\mathbb{S}^2$ or in $\mathbb{H}^2$ of constant width $w > 0$ (also $w < \frac{\pi}{2}$ in the spherical case), the area of K is at least the area of a Reuleaux triangle of diameter w.

Stability of the BLL-inequality

Theorem (Böröczky–S. 2022)

If $K \subset \mathcal{M}^2$ is a body of constant width $w > 0$ (also $w < \frac{\pi}{2}$ in the spherical case), $\varepsilon \geq 0$ and

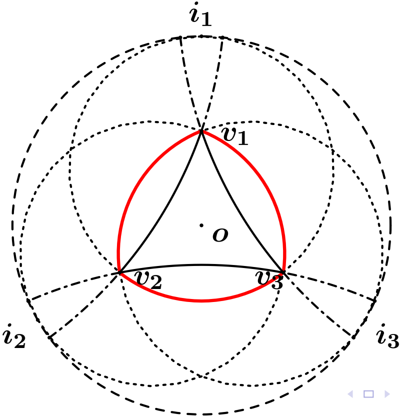
$$\text{area}\,(K) \leq (1+\varepsilon)\text{area}\,(U_w)\,,$$

then there exists a Reuleaux triangle $U \subset \mathcal{M}^2$ of width w such that $\delta_H(K, U) \leq \theta\varepsilon$ where $\theta > 0$ is an explicitly calculable constant depending on w and $\mathcal{M}^2$.

The h-convex isominwidth problem

Theorem (Böröczky, Freyer, S. 2024+)

Among h-convex bodies of fixed width, the horocyclic Reuleaux triangle has the smallest area.



Hyperbolic reduced polytopes

- In $\mathbb{H}^2$, polygons s. t. for every vertex v_i their projections t_i are interior points of the opposite side and $d(v_i, t_i) = w$ are reduced
- These are called *ordinary reduced polygons*
- Not all reduced polygons are ordinary (e.g. “long” rhombi)
- There are centrally symmetric reduced crosspolytopes in $\mathbb{H}^n$

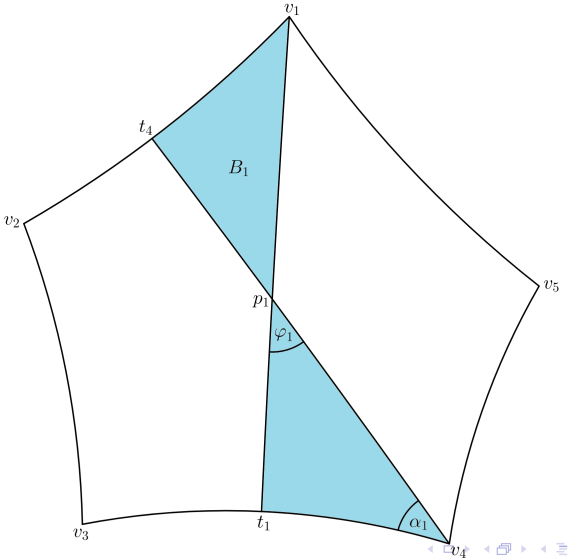
A few of Lassak's questions

Q1 Do regular $(2k+1)$-gons minimize/maximize the area?

Q2 Do regular $(2k+1)$-gons minimize/maximize the perimeter?

Q3 What is the smallest upperbound for the circumradii?

The key idea to tackle Q1 and Q2



Some remarks on the butterflies

- The sum of the angles φ_i is π
- The two "wings" of the butterflies are congruent for each B_i
- The butterflies cover the polygon
- One can measure the area and the perimeter with functions of φ_i
- We study the convexity of these functions

About the area

- $f_w(x) = \operatorname{arsinh} \frac{x\sqrt{1-\tanh^2 w}}{\tanh w - x}$,
- $g_w(x) = \frac{1+\cos x - \sqrt{(1+\cos x)^2 - 4\tanh^2 w \cos x}}{2\tanh w}$

Theorem (S. 2024+)

$$\operatorname{area}(P) = (n-2)\pi - 2\sum_{i=1}^{n} f_w(g_w(\varphi_i))$$

Theorem (S. 2024+)

The regular n-gon has the greatest area among ordinary reduced n-gons.

About the perimeter

- $g_w(x) = \frac{1+\cos x - \sqrt{(1+\cos x)^2 - 4\tanh^2 w \cos x}}{2\tanh w}$,
- $p_w(x) = \operatorname{arcosh}\left(\frac{1 - g_w(x)\tanh w}{\sqrt{1-\tanh^2 w}}\right)$

Theorem (S. 2024+)

$$\operatorname{perim}(P) = 2\sum_{i=1}^{n} p_w(\varphi_i)$$

Theorem (S. 2024+)

The regular n-gon has the smallest perimeter among ordinary reduced n-gons.

The diameter of ordinary reduced polygons

Theorem (Lassak 2024+)

$$\mathbf{diam}\,(P) < \mathbf{arcosh}\left(\mathbf{cosh}\,w\sqrt{1+\frac{\sqrt{2}}{2}\,\mathbf{sinh}\,w}\right)$$

The diameter of ordinary reduced polygons

Theorem (Lassak 2024+)

$$\mathrm{diam}\,(P) < \mathrm{arcosh}\left(\cosh w \sqrt{1 + \frac{\sqrt{2}}{2}\sinh w}\right)$$

Theorem (S. 2024+)

$$\mathrm{diam}\,(P) \leq 2\mathrm{arcosh}\left(\frac{\cosh w + \sqrt{\cosh^2 w + 8}}{4}\right)$$

with equality if and only if P is a regular triangle.

About the circumradius and inradius

Theorem (S. 2024+)

$$R\left(P\right) \leq \operatorname{arsinh}\left(\frac{2}{\sqrt{3}}\sqrt{\left(\frac{\cosh w + \sqrt{\cosh^2 w + 8}}{4}\right)^2 - 1}\right)$$

Theorem (S. 2024+)

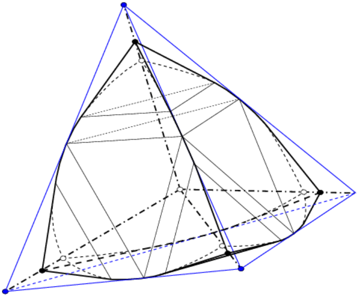
There is a boundary point $z \in \partial P$ s. t. $P \subset B\left(z, w\right)$.

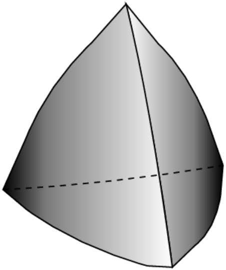
Pál's problem in $\mathbb{R}^3$

Theorem (Campi–Colesanti–Gronchi 1996)

Among rotationally symmetric bodies of thickness w in $\mathbb{R}^3$, the rotation of the regular triangle of height w has the smallest volume.

- It is conjectured that among convex bodies in $\mathbb{R}^3$ the Heil body has the smallest volume





The Blaschke–Lebesgue problem in $\mathbb{R}^3$

Theorem (Campi–Colesanti–Gronchi 1996)

Among rotationally symmetric bodies of constant width in $\mathbb{R}^3$, the rotation of the Reuleaux triangle has the smallest volume.

It is conjectured that the Meissner body has the smallest volume among bodies of constant width in $\mathbb{R}^3$.