The Illumination of Symmetric Cap Bodies

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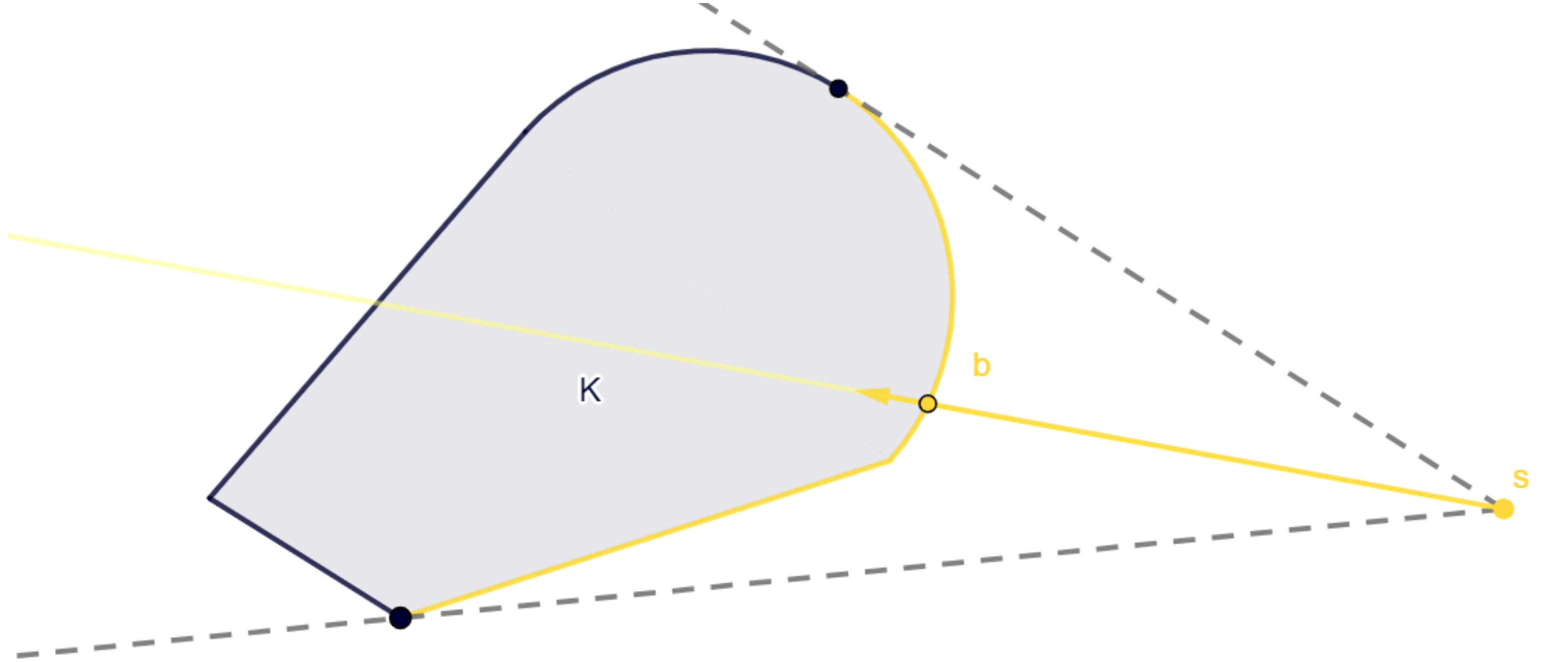
Introduction to Illumination

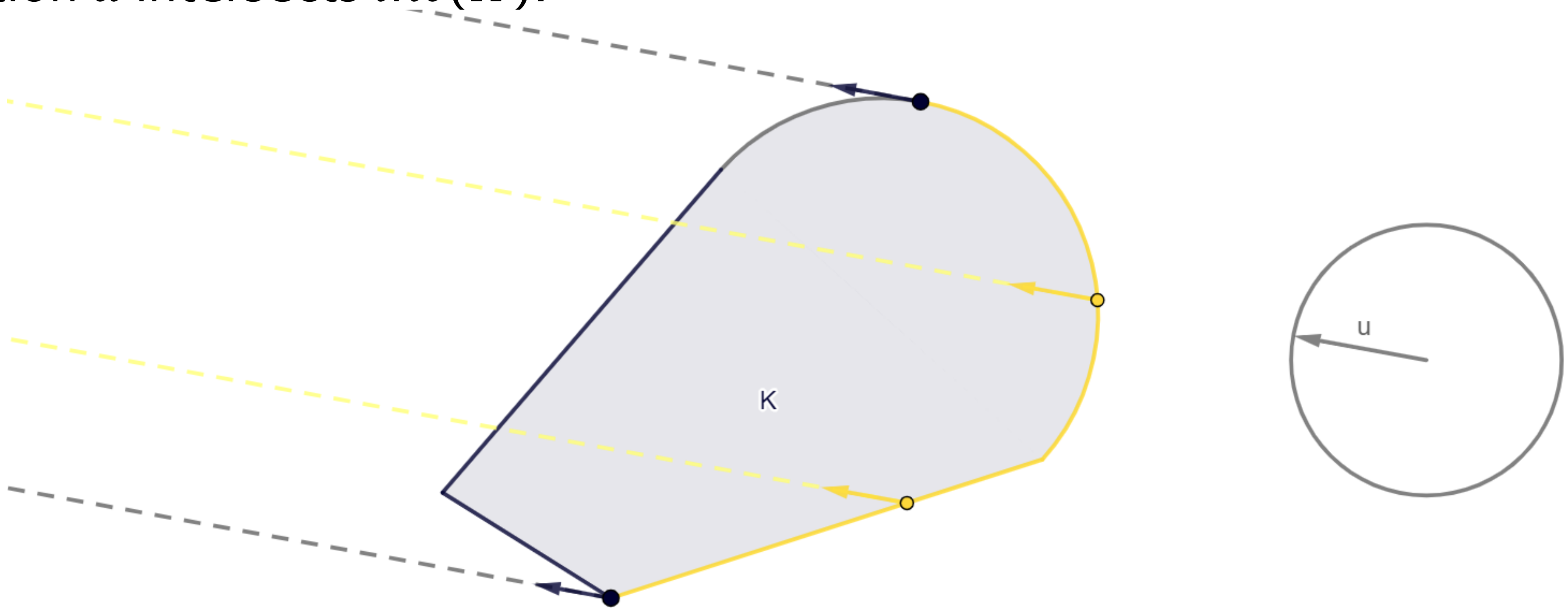
- A **convex body** $K \subseteq \mathbb{R}^d$, is a convex, compact set, with non-empty interior.
- $b \in bd(K)$ is **illuminated by $\boldsymbol{s}$** $\in \mathbb{R}^d \setminus K$, if the ray emanating from s going through b intersects $int(K)$ after it passes b.



- **A set** $\{s_1, s_2, \dots, s_n\} \subseteq \mathbb{R}^d \setminus K$ **illuminates a convex body** if every boundary point of K is illuminated by at least one element in $\{s_1, s_2, \dots, s_n\}$.

Equivalent Formulations of Illumination

- A direction $u \in \mathbb{S}^{d-1}$ **illuminates** $b \in bd(K)$, if the ray emanating from b in the direction u intersects $int(K)$.



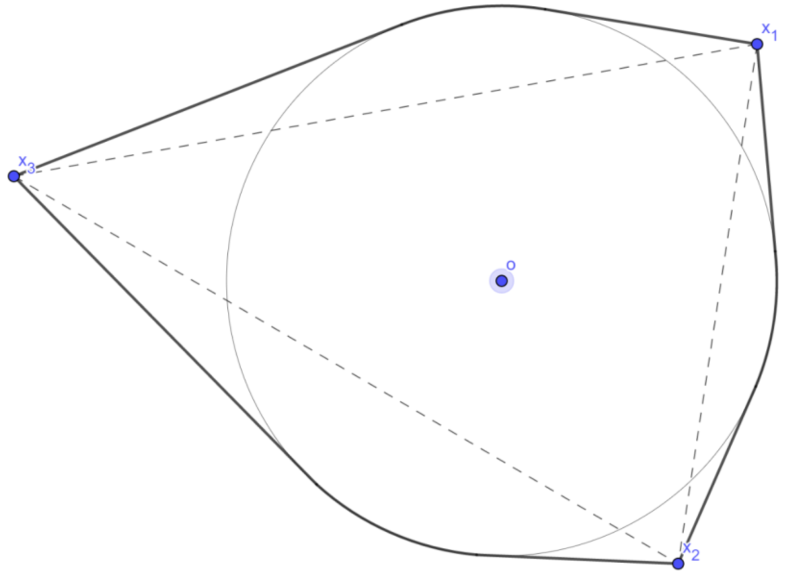
- The **illumination number** of K, denoted $I(K)$, equals the minimum number of directions/light sources needed to illuminate K.
- $I(K)$ is also equal to the minimum number of translates of $int(K)$ required to cover K.

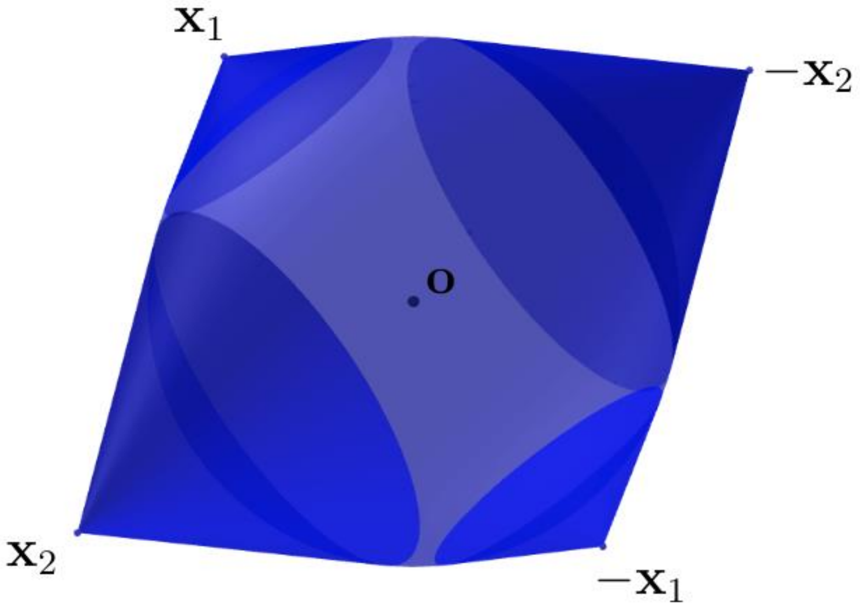
The Illumination Conjecture

- **Conjecture:** For any convex body $K \subseteq \mathbb{R}^d$, $I(K) \leq 2^d$. Furthermore, $I(K) = 2^d$ if and only if K is an affine image of the d-cube.
- Proven in $\mathbb{R}^2$ ([1], [2], [3] and [4]).
- In $\mathbb{R}^3$ has been proven for convex bodies with central symmetry [5], symmetry about a plane [6], and polytopes with affine symmetry [7].
- General bounds for $\mathbb{R}^3$, $\mathbb{R}^4$, $\mathbb{R}^5$, and $\mathbb{R}^6$, are $I(K) \leq 14, 96, 1091$, and 15373 [8] and [9].
- Rogers [10] proved in $\mathbb{R}^d$ that $I(K) \leq O\left(4^d\sqrt{d}\ln(d)\right)$ and if K is centrally symmetric $I(K) \leq O\left(2^d d\ln(d)\right)$ [10].

Cap Bodies

- A **cap body of a ball** $C \subseteq \mathbb{R}^d$, is the convex hull of the Euclidean ball B^d, and a finite set $\{x_1, x_2, \ldots, x_n\} \subseteq \mathbb{R}^d \backslash B^d$, such that $conv(\{x_i, x_j\}) \cap B^d \neq \emptyset$ for all $i \neq j$.





- Naszódi [11] proved that for any $\varepsilon > 0$ and all sufficiently large d, there exists a centrally symmetric cap body C, within an ε-region of B^d, with an exponential illumination number in terms of d.

Our Result

- We prove that for a centrally symmetric cap body $C \subseteq \mathbb{R}^d$ for $d \geq 3$,

$$I(C) \leq 2 + N_{\mathbb{S}^{d-2}}\left[\frac{\pi}{4}\right].$$

Where $N_{\mathbb{S}^{d-1}}[\alpha]$ denotes the minimum number of closed spherical caps with angular radius α required to cover $\mathbb{S}^{d-1}$.

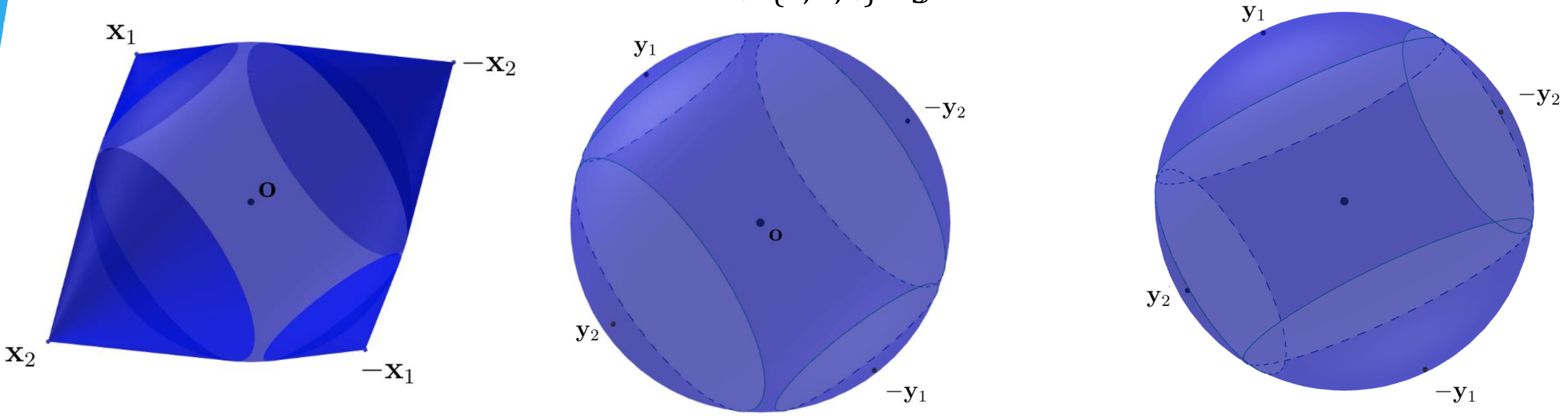
- Using the known bounds for $N_{\mathbb{S}^{d-2}}\left[\frac{\pi}{4}\right]$,

implies $I(C) \leq 6$ for $d = 3$, $I(C) \leq 12$ for $d = 4$ using [13],

and $I(C) < 2^d$ for $d \geq 20$ [12] (combined with [11]).

How to Illuminate Cap Bodies

- Every cap body $C = conv(B^d \cup \{x_1, x_2, \dots, x_n\})$ can be uniquely associated with a packing of closed spherical caps $\bigcup_{i \in \{1,\dots,n\}} C_{\mathbb{S}^{d-1}}[y_i, \alpha_i] \subseteq \mathbb{S}^{d-1}$.



- **Lemma**: A set of directions $\{u_1, u_2, \dots, u_k\} \subseteq \mathbb{S}^{d-1}$ illuminates C if and only if the set of open hemispheres centered at $-u_j$ for $j \in \{1,2, \dots, k\}$ separates the packing of spherical caps and covers $\mathbb{S}^{d-1}$.
- **Lemma**: A set of directions $\{u_1, u_2, \dots, u_k\} \subseteq \mathbb{S}^{d-1}$ illuminates C if and only if they positively span $\mathbb{R}^d$ and $\{-u_1, -u_2, \dots, -u_k\} \cap C_{\mathbb{S}^{d-1}}\left(y_i, \frac{\pi}{2} - \alpha_i\right) \neq \emptyset,\ \forall i$.

Lemma

- Let $\bigcup_{i\in\{1,2,\ldots,n\}} C_{\mathbb{S}^{d-1}}\left(\pm y_i, \frac{\pi}{2}-\alpha_i\right)$ be the set of open spherical caps associated to a centrally symmetric cap body C.
- **Lemma**: $C_{\mathbb{S}^{d-1}}\left[\pm y_i, \frac{\pi}{2}-\alpha_i\right] \cap C_{\mathbb{S}^{d-1}}\left[\pm y_j, \frac{\pi}{2}-\alpha_j\right] \neq \emptyset$.

Proof: It suffices to show $C_{\mathbb{S}^{d-1}}\left[y_i, \frac{\pi}{2}-\alpha_i\right] \cap C_{\mathbb{S}^{d-1}}\left[-y_j, \frac{\pi}{2}-\alpha_j\right] \neq \emptyset$.

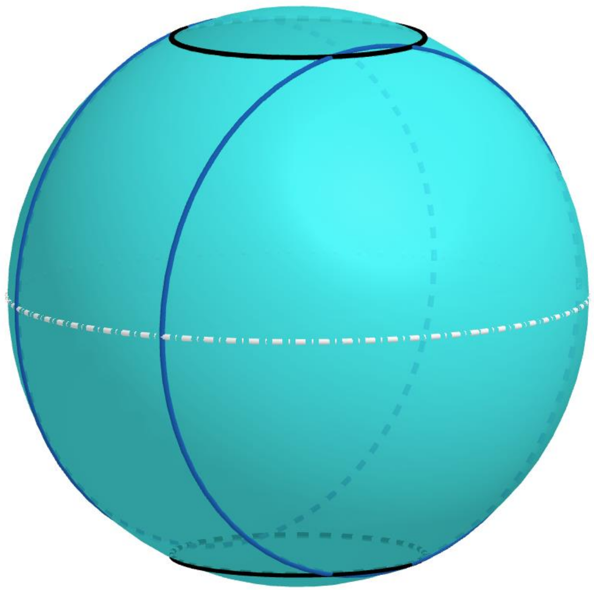
Let H be the hyperplane passing through the origin separating $C_{\mathbb{S}^{d-1}}[y_i, \alpha_i]$ and $C_{\mathbb{S}^{d-1}}[y_j, \alpha_j]$ with a normal vector n_{ij}. Then $\angle(n_{ij}, y_i) \leq \frac{\pi}{2}-\alpha_i$ and $\angle(-n_{ij}, y_j) \leq \frac{\pi}{2}-\alpha_j$. The latter implies $\angle(n_{ij}, -y_j) \leq \frac{\pi}{2}-\alpha_j$ so

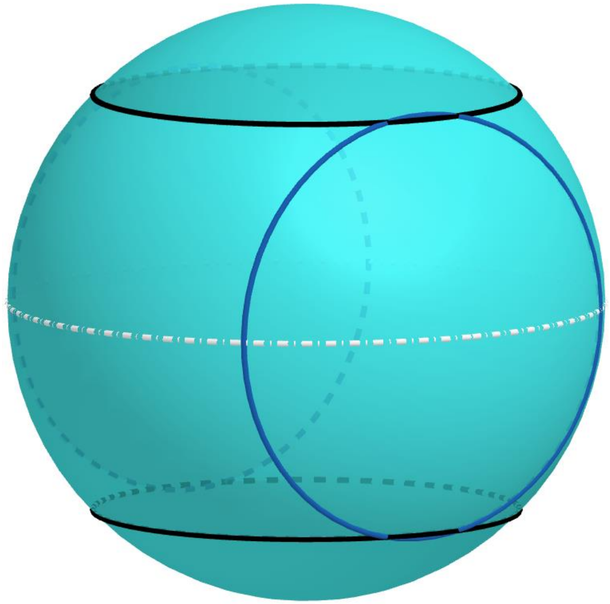
$$n_{ij} \in C_{\mathbb{S}^{d-1}}\left[y_i, \frac{\pi}{2}-\alpha_i\right] \cap C_{\mathbb{S}^{d-1}}\left[-y_j, \frac{\pi}{2}-\alpha_j\right].$$

Another Lemma

- Need to find a set of size $2 + N_{\mathbb{S}^{d-2}}\left[\frac{\pi}{4}\right]$ that positively spans $\mathbb{R}^d$, and pierces the set of open spherical caps $\bigcup_{i\in\{1,2,\dots,n\}} C_{\mathbb{S}^{d-1}}(\pm y_i, \beta_i\ )$. Here $\beta_i = \frac{\pi}{2} - \alpha_i$.
- **Lemma**: Suppose $0 < \beta_1 \le \beta_2 \le \cdots \le \beta_n < \frac{\pi}{2}$, orient $\pm y_1$ at the north and south pole and let $\mathbb{S}^{d-2}$ be the equator, then for $i \in \{2,3,\dots,n\}$:

$\mathbb{S}^{d-2} \cap C_{\mathbb{S}^{d-1}}[\pm y_i, \beta_i]$ is a closed $(d-2)$-spherical cap with a radius of at least $\frac{\pi}{4}$.





Proof of Theorem

- Now cover $\mathbb{S}^{d-2}$ with a minimal number of closed spherical caps having radii $\frac{\pi}{4}$, and take $\{u_1, u_2, \dots, u_{N_{\mathbb{S}^{d-2}}\left[\frac{\pi}{4}\right]}\}$ as the set of center points.
- No point in $\mathbb{S}^{d-2}$ is at a distance at least $\frac{\pi}{4}$ from this set, which implies $\{u_1, u_2, \dots, u_{N_{\mathbb{S}^{d-2}}\left[\frac{\pi}{4}\right]}\}$ pierces the caps $C_{\mathbb{S}^{d-1}}[\pm y_i, \beta_i]$ for $i \in \{2, 3, \dots, n\}$.
- With a proper chosen isometry of $\{u_1, u_2, \dots, u_{N_{\mathbb{S}^{d-2}}\left[\frac{\pi}{4}\right]}\}$ we can ensure they pierce the open spherical caps.
- Now taking this set along with $\pm y_1$ gives us a set that pierces every cap.
- This set of points positively spans $\mathbb{R}^d$ as no open hemisphere is left unpierced.

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