Point-hyperplane incidences via extremal graph theory

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joint work with Aleksa Milojević and Benny Sudakov

Szemerédi-Trotter theorem (1983)

Let P be a set of m points and L be a set of n lines in $\mathbb{R}^2$. Then

$$I(P, L) = O(m^{2/3}n^{2/3} + m + n).$$

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Let P be a set of m points and L be a set of n lines in $\mathbb{F}^2$, $m \geq n$. Then

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Lower bound: choose p prime $p \approx \sqrt{m}$
$P = \mathbb{F}_p^2$ and L arbitrary set of n lines.

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Problem (Chazelle 1993)

What is the maximum number of incidences between m points and n hyperplanes in $\mathbb{R}^d$, assuming the incidence graph is $\mathbf{K_{s,s}}$**-free**?

Theorem (Apfelbaum-Sharir)

If P is a set of m points, and H is a set of n hyperplanes in $\mathbb{R}^d$ such that the incidence graph is $K_{s,s}$-free, then

$$I(P,H) = O_s((mn)^{1-\frac{1}{d+1}} + m + n).$$

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Lower bound (Sudakov, T. 2023): there exists $s = s(d)$ such that

$$I(P,H) \gtrapprox \begin{cases} (mn)^{1-\frac{2d+3}{(d+2)(d+3)}} & \text{if } d \text{ is odd,} \\ (mn)^{1-\frac{2d^2+d-2}{(d+2)(d^2+2d-2)}} & \text{if } d \text{ is even.} \end{cases}$$

Theorem (Milojević, Sudakov, T. 2024+)

Let P be a set of m points, H a set of n hyperplanes in $\mathbb{F}^d$, $n = m^\alpha$, such that the incidence graph is $K_{s,s}$-free. Then

$$I(P,H) \leq \begin{cases} O_{s,d}(m) & \text{if } \alpha \in (0, \frac{1}{d}], \\ O_{s,d}(m^{1-\frac{1}{d+2-t}}n) & \text{if } \alpha \in [\frac{t-1}{d+2-t}, \frac{t}{d+2-t}],\ t \in \{2,\dots,d\}, \\ O_{s,d}(mn^{1-\frac{1}{t}}) & \text{if } \alpha \in [\frac{t}{d+2-t}, \frac{t}{d+1-t}]\ t \in \{2,\dots,d\}, \\ O_{s,d}(n) & \text{if } \alpha \in [d, \infty). \end{cases}$$

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This is **sharp**: for every m, n, there exists a field $\mathbb{F} = \mathbb{F}(d, m, n)$ and a set of points and hyperplanes achieving this bound.

Corollary

Let P be a set of m points, H a set of n hyperplanes in $\mathbb{F}^d$, such that the incidence graph is $K_{s,s}$-free. Then

$$I(P, H) \leq O_{s,d}((mn)^{1-\frac{1}{d+2}} + m + n).$$

Moreover, if $m = n^{\frac{t}{d+1-t}}$ for some integer $t \in \{2, \ldots, d\}$, then

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If $m = n$ and d is odd, same bound as Apfelbaum-Sharir!

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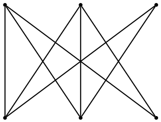
P_3



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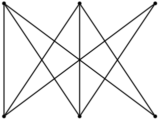
P_3 

Lemma

Let G be a bipartite graph with $n + n$ vertices with no induced P_3, and no $K_{s,s}$. Then

$$e(G) = O_s(n^{3/2}).$$

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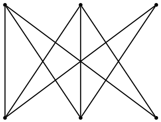
P_3 

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Idea: **dependent random choice**

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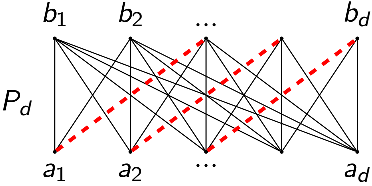

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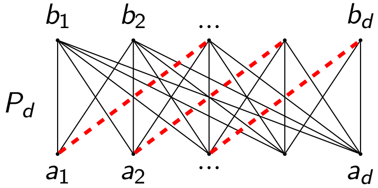

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There exists $s = s(d)$ such that for every prime p, there is a (k, s)-subspace evasive set in $\mathbb{F}_p^d$ of size p^{d-k}.

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- H is the set of all hyperplanes with normalvector in N.

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Theorem (Apfelbaum and Sharir)

Given m points and n hyperplanes in $\mathbb{R}^d$, $m \leq n$, with $I = \varepsilon mn$ incidences, $\varepsilon > n^{-1/(d-1)}$, the incidence graph contains a complete bipartite graph of size $\Omega(\epsilon^{d-1}mn)$.

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$$\Omega_d(\varepsilon^{d-1}mn) \text{ if } \varepsilon > 100 \max\{m^{-\frac{1}{d-1}}, n^{-\frac{1}{d}}\}$$
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These bounds are **sharp**.