The Honeycomb Conjecture in normed planes

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PRELIMINARIES

1. A convex *mosaic* or *tiling* $\mathcal{T}$ of $\mathbb{R}^2$ is a family of mutually nonoverlapping convex disk, called cells or tiles, with the property that $\bigcup \mathcal{T} = \mathbb{R}^2$. A convex tiling is normal if for some $0 < \hat{r} < \hat{R}$, every cell contains a Euclidean disk of radius $\hat{r}$ and is contained in a Euclidean disk of radius $\hat{R}$.
2. A convex tiling is called *edge-to-edge*, if every edge of a cell belongs to exactly one more cell.
3. B^2 : Closed Euclidean unit disk centered at o.

MOTIVATION

CONJECTURE (HONEYCOMB CONJECTURE, VARRO)

In a decomposition of the Euclidean plane into cells of unit area, the average perimeter of the cells is minimal for the regular hexagonal tiling.

1. In the 1940s, L. Fejes Tóth proved for normal, convex tilings.
2. In the 2000s, Hales dropped the condition of convexity.

QUESTION

For a normed plane $\mathcal{M}$, Is it true that a tiling of $\mathcal{M}$ with unit area tiles, the average perimeter of a cell is minimal for a hexagonal tiling?

Motivation

Conjecture (Honeycomb Conjecture, Varro)

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PRELIMINARIES

1. Every origin-symmetric convex disk M is the unit disk of a normed plane; and the unit disk of a normed plane is an origin-symmetric convex disk.
2. M-Perimeter of a convex disk K: supremum of the total edge lengths of the convex polygons (measured in the norm of $\mathcal{M}$) inscribed in K, denoted by $\mathrm{perim}_M(K)$.
3. Every 'meaning' definition of area is a scalar multiple of Euclidean area. We assume it is Euclidean area, denoted by $\mathrm{area}(\cdot)$.

PRELIMINARIES

DEFINITION

Let $\mathcal{T}$ be a convex, normal tiling in the normed plane $\mathcal{M}$. For any $R > 0$, let $\mathcal{T}(R)$ denote the family of cells of $\mathcal{T}$ contained in $R\mathbf{B}^2$. Let $\alpha > 0$. We define the *lower average αth powered perimeter* of a cell of $\mathcal{T}$ as the quantity

$$\underline{P}_\alpha(\mathcal{T}) = \liminf_{R\to\infty} \frac{\sum_{C\in\mathcal{T}(R)} (\mathrm{perim}_M(C))^\alpha}{\mathrm{card}(\mathcal{T}(R))}.$$

Similarly We define the *upper average αth powered perimeter* of a cell of $\mathcal{T}$, denoted by $\overline{P}_\alpha(\mathcal{T})$, replacing $\liminf$ by $\limsup$. If $\underline{P}_\alpha(\mathcal{T}) = \overline{P}_\alpha(\mathcal{T})$, we call this quantity the *average αth powered perimeter* of a cell of $\mathcal{T}$, and denote it by $P_\alpha(\mathcal{T})$.

PRELIMINARIES

1. If $\alpha = 1$, we omit it from the notation, and called the corresponding quantities the lower/upper/- average perimeter.
2. We define the quantities $\underline{P}_{\log}(\mathcal{T}), \overline{P}_{\log}(\mathcal{T})$ and $P_{\log}(\mathcal{T})$ similarly, replacing $(\mathrm{perim}_M(C))^\alpha$ by $\log(\mathrm{perim}_M(C))$ in the above definitions.

A weaker version of Honeycomb Conjecture in any normed planes

Theorem (Lángi, Wang)

For any normed plane $\mathcal{M}$ there is a hexagonal tiling $\mathcal{T}_{hex}$ of $\mathcal{M}$ such that for any convex, normal tiling $\mathcal{T}$ of $\mathcal{M}$, we have

$$\underline{P}_2(\mathcal{T}) \geq P_2(\mathcal{T}_{hex}).$$

Remark

For any $\alpha, \beta \in (0, \infty)$ with $\alpha < \beta$ and any normal, convex tiling $\mathcal{T}$ in $\mathcal{M}$, we have $\exp(\underline{P}_{\log}(\mathcal{T})) \leq (\underline{P}_\alpha(\mathcal{T}))^{1/\alpha} \leq (\underline{P}_\beta(\mathcal{T}))^{1/\beta}$ and $\exp(\overline{P}_{\log}(\mathcal{T})) \leq (\overline{P}_\alpha(\mathcal{T}))^{1/\alpha} \leq (\overline{P}_\beta(\mathcal{T}))^{1/\beta}$. Furthermore, if $\mathcal{T}$ is a hexagonal tiling, we have equality in all the previous inequalities.

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Sketch proof

Theorem (Busemann)

Let $\mathcal{M}$ be a normed plane. The area enclosed by a simple, closed curve Γ of a given M-length is maximized if Γ is the boundary of a plane convex body K homothetic to the so-called isoperimetrix M_{iso} of $\mathcal{M}$, obtained as the polar of the rotated copy of the unit disk M of $\mathcal{M}$ by $\frac{\pi}{2}$ (see Figure 1).

SKETCH PROOF

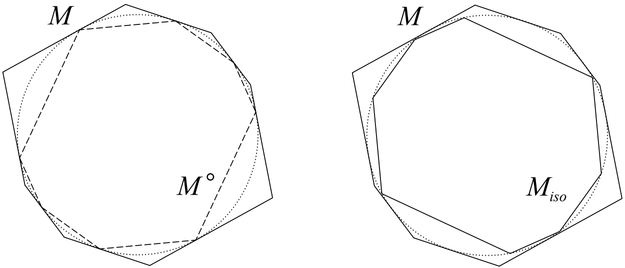

FIGURE: The isoperimetrix M_{iso} of a norm with unit disk M. The dotted circle is the Euclidean unit disk $\mathbf{B}^2$ centered at o. The left-hand side panel shows M and its polar M°, the isoperimetrix in the righ-hand side panel is a rotated copy of M° by $\frac{\pi}{2}$

Sketch proof

Theorem (Chakerian)

Let $\mathcal{M}$ be a normed plane with unit disk M. Let the isoperimetrix of the plane be M_{iso}. Let K be an arbitrary convex n-gon in $\mathcal{M}$, and let K^* be the convex n-gon circumscribed about M_{iso} whose sides have the same outer unit normals as the sides of K. Let the M-perimeter of K be L, the area of K be F, and the area of K^* be f. Then

$$L^2 - 4fF \geq 0,$$

with equality if and only if K is homothetic to K^*.

SKETCH PROOF

THEOREM (DOWKER)

For any convex disk K in $\mathbb{R}^2$, let

$$A_K(n) = \inf\{\text{area}(P) : P \text{ is a convex } n\text{-gon circumscribed about } K\}.$$

Then the sequence $\{A_K(n)\}$ is convex. In other words, for any $n \geq 4$, we have

$$A_K(n-1) + A_K(n+1) \geq 2A_K(n).$$

1. $v(C)$: number of sides of the cell C of the convex, normal tiling $\mathcal{T}$.
2. $\mathcal{T}(R)$: family of cells of $\mathcal{T}$ in $R\mathbf{B}^2$ of radius R.

Sketch proof

Theorem (Dowker)

Let K be an o-symmetric plane convex body. Then, for every $m \geq 2$, there is a centrally symmetric convex $(2m)$-gon P circumscribed about K with $\mathrm{area}(P) = A_K(2m)$.

Remark

The upper average number of sides in any normal, convex tiling is at most 6.

Sketch proof

For any $C \in \mathcal{T}(R)$, let C^* denote the convex polygon circumscribed about M_{iso} such that the sides of C and C^* have the same outer unit normals, and let $v(C)$ denote the number of sides of C. Then:

$$\begin{aligned} \underline{P}_2(\mathcal{T}) &= \liminf_{R\to\infty} \frac{\sum_{C\in\mathcal{T}(R)} (\mathrm{perim}_M(C))^2}{\mathrm{card}(\mathcal{T}(R))} \\ &\geq 4 \liminf_{R\to\infty} \frac{\sum_{C\in\mathcal{T}(R)} (\mathrm{area}(C^*))}{\mathrm{card}\,\mathcal{T}(R)} \\ &\geq 4 \liminf_{R\to\infty} \frac{\sum_{C\in\mathcal{T}(R)} (A_{M_{iso}}(v(C)))}{\mathrm{card}\,\mathcal{T}(R)} \geq 4A_{M_{iso}}(6) \end{aligned}$$

This is attained by a hexagonal tiling.

Related results

Definition

Let $\alpha \in (0, \infty)$. We say that the normed plane $\mathcal{M}$ satisfies the α-*honeycomb property*, if there is a hexagonal tiling $\mathcal{T}_{hex}$ of $\mathcal{M}$ such that for any convex, normal tiling $\mathcal{T}$ of $\mathcal{M}$, we have

$$\underline{P}_\alpha(\mathcal{T}) \geq P_\alpha(\mathcal{T}_{hex}).$$

Similarly, we say that it satisfies the *log-honeycomb* (or 0-*honeycomb*) *property* if the same holds for the lower average log-perimeter of a cell of $\mathcal{T}$.

Related results

Definition

Let $\alpha \in (0, \infty)$. We say that a convex disk K satisfies the α*-Dowker property* if the sequence $\{A_K^\alpha(n)\}$ is convex. Furthermore, we say that it satisfies the *log-Dowker* (or 0*-Dowker*) *property* if the sequence $\{\log A_K(n)\}$ is convex.

Related results

Definition

Let $\alpha \in (0, \infty)$. We say that a convex disk K satisfies the *weak α-Dowker property* if

$$\frac{n-6}{n-m} A_K^{\alpha}(m) + \frac{6-m}{n-m} A_K^{\alpha}(n) \geq A_K^{\alpha}(6)$$

holds for any $3 \leq m < 6 < n$. Similarly, we say that K satisfies the *weak log-Dowker* (or *weak* 0*-Dowker*) *property* if

$$\frac{n-6}{n-m} \log A_K(m) + \frac{6-m}{n-m} \log A_K(n) \geq \log A_K(6)$$

holds for any $3 \leq m < 6 < n$.

RELATED RESULTS

THEOREM (LÁNGI, WANG)

Let $\mathcal{M}$ be a normed splane. For any $\alpha \in (0, \infty)$, if the isoperimetrix M_{iso} of $\mathcal{M}$ satisfies the weak α-Dowker property, then the normed plane $\mathcal{M}$ satisfies the (2α)-honeycomb property.

QUESTION

Which convex disks satisfy the (weak) $\frac{1}{2}$-Dowker property?

REMARK

It is an elementary exercise to check that $A_{\mathbf{B}^2}(n) = n \tan \frac{\pi}{n}$, implying that $\mathbf{B}^2$ satisfies the log-Dowker property, and the Euclidean plane satisfies the log-honeycomb property.

RSULTS ABOUT POLYGONAL NORMS

THEOREM

If the unit disk of $\mathcal{M}$ is a convex $(2k)$-gon and $\alpha \in (0, \infty)$, then there is an algorithm that checks in $\mathcal{O}(k^3 \log^2 k)$ steps if M_{iso} satisfies the (weak) α-Dowker property or not.

THEOREM (LÁNGI, WANG)

A regular $(2k)$-gon P_k, with $k \geq 2$, satisfies the weak $\frac{1}{2}$-Dowker property if and only if $k \neq 4, 5, 7$.

THEOREM (LÁNGI, WANG)

If the unit disk of a normed plane $\mathcal{M}$ is a regular $(2k)$-gon with $k \neq 4, 5, 7$, then $\mathcal{M}$ satisfies the honeycomb property.

Results about polygonal norms

Remark

If $k \geq 4$, then $A_{2k-2}(P_k)$ is a convex combination of $A_{2k-1}(P_k)$ and $A_{2k-3}(P_k)$, implying that in this case P_k does not satisfy the α-Dowker property for any $\alpha < 1$.

Results about general norms

Theorem (Lángi, Wang, 2024)

If K is a convex disk in $\mathbb{R}^2$ with C^4-class boundary and strictly positive curvature everywhere, then there is some value $n(K) \in \mathbb{R}$ such that for any $n \geq n(K)$, we have

$$\log A_K(n-1) + \log A_K(n+1) \geq 2 \log A_K(n).$$

Theorem (Lángi, Wang)

Let K be smooth and strictly convex. Then there is some value $\alpha < 1$ such that K satisfies the weak α-Dowker property.

Results about general norms

Recall that $A_{\mathbf{B}^2}(n) = n \tan \frac{\pi}{n}$. In the following theorem, we let

$$\varepsilon_0 = \frac{\sqrt{A_{\mathbf{B}^2}(5)} + \sqrt{A_{\mathbf{B}^2}(7)} - 2\sqrt{A_{\mathbf{B}^2}(6)}}{\sqrt{A_{\mathbf{B}^2}(5)} + \sqrt{A_{\mathbf{B}^2}(7)} + 2\sqrt{A_{\mathbf{B}^2}(6)}} = 0.002623\ldots,$$

and denote the Hausdorff distance of the convex bodies K, L by $d_H(K, L)$.

Theorem (Lángi, Wang)

Let $\mathcal{M}$ be a normed plane with unit disk M, and assume that $d_H(M, \mathbf{B}^2) \le \varepsilon_0$. Then $\mathcal{M}$ satisfies the honeycomb property.

A conjecture of Steinhaus

Definition

Let $\mathcal{T}$ be a tiling of a normed plane $\mathcal{M}$. Let $\mathcal{T}(R)$ denote the family of cells of $\mathcal{T}$ in $R\mathbf{B}^2$. Then the *lower average isoperimetric ratio* of a cell of $\mathcal{T}$ is defined as

$$\underline{I}(\mathcal{T}) = \liminf_{R\to\infty} \frac{\sum_{C\in\mathcal{T}(R)} \frac{\mathrm{perim}_M(C)^2}{\mathrm{area}(C)}}{\mathrm{card}(\mathcal{T}(C))}.$$

If we replace the $\liminf$ in the above definition by $\limsup$, we obtain the *upper average isoperimetric ratio* $\bar{I}(\mathcal{T})$ of a cell. If these quantities are equal, the common value is called the *average isoperimetric ratio* of a cell, denoted by $I(\mathcal{T})$.

A conjecture of Steinhaus

Conjecture (Steinhaus)

For any tiling $\mathcal{T}$ in the Euclidean plane with tiles whose diameters are at least D for some fixed $D > 0$, the maximum isoperimetric ratio $\frac{\text{perim}(C)^2}{\text{area}(C)}$ of the cells C of $\mathcal{T}$ is minimal if $\mathcal{T}$ is a regular hexagonal tiling.

Theorem (Lángi, Wang)

For any normed plane $\mathcal{M}$ there is a hexagonal tiling $\mathcal{T}_{hex}$ of $\mathcal{M}$ such that for any convex, normal tiling $\mathcal{T}$ of $\mathcal{M}$, we have

$$\underline{I}(\mathcal{T}) \geq I(\mathcal{T}_{hex}).$$

Furthermore, if $\mathcal{M}$ is a Euclidean plane, then $\mathcal{T}_{hex}$ is a regular hexagonal tiling.

Thank you!