Answers to questions of Grünbaum and Loewner (joint work with S. Myroshnychenko and K. Tatarko)

Vlad Yaskin

University of Alberta

yaskin@ualberta.ca

Budapest, July 2 - 5, 2024

Introduction

Let K be a convex body in $\mathbb{R}^n$. The centroid of K (also known as the center of mass or the barycenter) is the point

$$\frac{1}{|K|}\int_K x\,dx,$$

where $|K|$ denotes the volume of K.

Introduction

Problem (Grünbaum)

Is the centroid $c(K)$ of $K \subset \mathbb{R}^n$ the centroid of at least $n+1$ different $(n-1)$-dimensional sections of K through $c(K)$?

Introduction

History:

Consider the subset of the interior of K with the property that each point p from this set is the centroid of at least $n+1$ different $(n-1)$-dimensional sections of K through p. Grünbaum claimed that this set is non-empty for every convex body K and asked whether the centroid $c(K)$ of K belongs to this set.

History:

Consider the subset of the interior of K with the property that each point p from this set is the centroid of at least $n+1$ different $(n-1)$-dimensional sections of K through p. Grünbaum claimed that this set is non-empty for every convex body K and asked whether the centroid $c(K)$ of K belongs to this set.

Patáková, Tancer, and Wagner [2022] showed that one of the auxiliary statements in Grünbaum's paper is incorrect.

They also showed that every convex body $K \subset \mathbb{R}^n$, $n \geq 3$, contains a point p that is the centroid of at least four sections of K through p.

A related question was asked by Loewner.

Problem (Loewner)

Let $\mu(K)$ be the number of hyperplane sections of K passing through $c(K)$ whose centroid is the same as $c(K)$. Let

$$\mu(n) = \min_{K \in \mathcal{K}^n} \mu(K),$$

where $\mathcal{K}^n$ is the class of all convex bodies in $\mathbb{R}^n$. What is the value of $\mu(n)$?

Introduction

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- Grünbaum has shown that every point in the interior of a convex body is the centroid of at least one section. Thus $\mu(n) \geq 1$ for all $n \geq 3$.
- In this talk we will show that $\mu(n) = 1$ for $n \geq 5$. In particular, Grünbaum's question has a negative answer for $n \geq 5$. We construct a body of revolution K satisfying $\mu(K) = 1$ using Fourier analytic tools and exploiting the fact that there are origin-symmetric convex bodies that are not intersection bodies in $\mathbb{R}^n$ for $n \geq 5$.

We say that a compact set $K \subset \mathbb{R}^n$ is star-shaped about the origin 0 if for every point $x \in K$ each point of the interval $[0, x)$ is an interior point of K.

The Minkowski functional of K is defined by

$$\|x\|_K = \min\{\lambda \geq 0 : x \in \lambda K\}.$$

K is called a star body if it is compact, star-shaped about the origin and its Minkowski functional is a continuous function on $\mathbb{R}^n$.

The radial function of a star body K is defined by

$$\rho_K(\xi) = \max\{\lambda > 0 : \lambda\xi \in K\}, \quad \xi \in S^{n-1}.$$

Observe that $\rho_K(\xi) = \|\xi\|_K^{-1}$ for any $\xi \in S^{n-1}$.

We say that K is origin-symmetric if $x \in K \Leftrightarrow -x \in K$. For an origin-symmetric star body K we have $\rho_K(\xi) = \rho_K(-\xi)$ for all $\xi \in S^{n-1}$.

The notion of the intersection body of a star body was first introduced by Lutwak and has played an important role in convex geometry.

Let K and L be origin-symmetric star bodies in $\mathbb{R}^n$. We say that K is the intersection body of L if

$$\rho_K(\xi) = |L \cap \xi^\perp|, \quad \text{for all } \xi \in S^{n-1}$$

where $\xi^\perp = \{x \in \mathbb{R}^n : \langle x, \xi \rangle = 0\}$ is the hyperplane through the origin orthogonal to the vector ξ.

Introduction

Passing to polar coordinates, we see that

$$\rho_K(\xi) = \frac{1}{n-1} \int_{S^{n-1}\cap\xi^\perp} \rho_L^{n-1}(x)\, dx.$$

In terms of the Fourier transform this means

$$\|\xi\|_K^{-1} = \frac{1}{\pi(n-1)} \left(\|\cdot\|_L^{-n+1}\right)^\wedge(\xi),$$

which implies that

$$\left(\|\cdot\|_K^{-1}\right)^\wedge(\xi) = \frac{(2\pi)^n}{\pi(n-1)}\, \|\xi\|_L^{-n+1} > 0.$$

Introduction

A more general class of intersection bodies is defined as the closure of the class of intersection bodies of star bodies in the radial metric.

Fourier analytic characterization of intersection bodies (Koldobsky):
An origin-symmetric star body in $\mathbb{R}^n$ is an intersection body if and only if the Fourier transform of $\|\cdot\|_K^{-1}$ is a positive distribution.

All origin-symmetric convex bodies in dimensions 2, 3, and 4 are intersection bodies, while in dimensions $n \geq 5$ there are non-intersection bodies.

Introduction

Intersection bodies were a key ingredient in the solution of the celebrated Busemann–Petty problem; the connection was discovered by Lutwak.

Let K and L be origin-symmetric convex bodies such that

$$|K \cap \xi^{\perp}| \leq |L \cap \xi^{\perp}|, \qquad \forall \xi \in S^{n-1}.$$

Does this guarantee that $|K| \leq |L|$?

Answer:
Yes, if $n = 2, 3, 4$.
No, if $n \geq 5$.

Main result

Theorem (Myroshnychenko, Tatarko, Y.)

There exists a convex body $K \subset \mathbb{R}^n$, $n \geq 5$, with centroid at the origin, such that

$$\langle c(K \cap \xi^\perp), e_n \rangle > 0$$

for all $\xi \neq \pm e_n$.

Idea of the proof

We will construct K, a convex body of revolution about the x_n-axis such that

$$|K| \langle c(K), e_n \rangle = \int_K x_n \, dx = \frac{1}{n+1} \int_{S^{n-1}} x_n \rho_K^{n+1}(x) \, dx = 0,$$

and

$$|K \cap \xi^\perp| \langle c(K \cap \xi^\perp), e_n \rangle = \int_{K \cap \xi^\perp} x_n \, dx = \frac{1}{n} \int_{S^{n-1} \cap \xi^\perp} x_n \rho_K^n(x) \, dx > 0,$$

for every $\xi \neq \pm e_n$

Idea of proof

To satisfy the last condition, take any even function $g \in C^\infty(S^{n-1})$ such that

- g is rotationally invariant about the x_n-axis,
- $g(\xi) > 0$ for all $\xi \in S^{n-1} \setminus \{\pm e_n\}$,
- $g(\pm e_n) = 0$.

Also take a small $\varepsilon > 0$.

We want

$$\frac{1}{n}\int_{S^{n-1}\cap\xi^\perp} x_n \rho_K^n(x)\,dx = \frac{1}{2n}\int_{S^{n-1}\cap\xi^\perp} x_n\left(\rho_K^n(x) - \rho_K^n(-x)\right)\,dx = \varepsilon g(\xi)$$

Using the connection between the spherical Radon transform and the Fourier transform, we get

$$\frac{1}{2n}\int_{S^{n-1}\cap\xi^\perp} x_n\left(\rho_K^n(x)-\rho_K^n(-x)\right)\,dx$$

$$=\frac{1}{2n\pi}\left(x_n\left(\rho_K^n(x)-\rho_K^n(-x)\right)\right)^\wedge(\xi)=\varepsilon g(\xi)$$

Inverting the Fourier transform, we get the odd part of ρ_K^n:

$$\rho_K^n(x)-\rho_K^n(-x)=\varepsilon\frac{n(2\pi)^{1-n}}{x_n}\widehat{g}(x)=:\varepsilon\phi(x).$$

(We extend g to a homogenous function on $\mathbb{R}^n\setminus\{0\}$ of degree -1).

Idea of proof

As the even part of ρ_K^n we take ρ_M^n, where M is a convex body (with positive curvature) of revolution about the x_n-axis, which is a non-intersection body.

Thus K is constructed as follows:

$$\rho_K^n(\xi) = \rho_M^n(\xi) + \varepsilon\phi(\xi), \qquad \xi \in S^{n-1}.$$

Since M has positive curvature, K will be convex for small enough $\varepsilon > 0$.

Proof

We will now show how to satisfy the condition $C(K) = 0$.

Since K is a body of revolution, it is enough to ensure that the quantity below is zero.

$$
\begin{aligned}
|K| \langle c(K), e_n \rangle &= \int_K x_n \, dx = \frac{1}{n+1} \int_{S^{n-1}} x_n \rho_K^{n+1}(x) \, dx \\
&= \frac{1}{2(n+1)} \int_{S^{n-1}} x_n \left(\rho_K^{n+1}(x) - \rho_K^{n+1}(-x) \right) \, dx \\
&= \frac{1}{2(n+1)} \int_{S^{n-1}} \left(\frac{\rho_K^{n+1}(x) - \rho_K^{n+1}(-x)}{\varepsilon \phi(x)} \right) x_n \varepsilon \phi(x) \, dx.
\end{aligned}
$$

Idea of proof

Since ϕ is odd and M is origin-symmetric, we have

$$\begin{aligned}
&\frac{\rho_K^{n+1}(x) - \rho_K^{n+1}(-x)}{\varepsilon\phi(x)} \\
&= \frac{(\rho_M^n(x) + \varepsilon\phi(x))^{\frac{n+1}{n}} - (\rho_M^n(-x) + \varepsilon\phi(-x))^{\frac{n+1}{n}}}{\varepsilon\phi(x)} \\
&= \frac{(\rho_M^n(x) + \varepsilon\phi(x))^{\frac{n+1}{n}} - (\rho_M^n(x) - \varepsilon\phi(x))^{\frac{n+1}{n}}}{\varepsilon\phi(x)} \\
&= \rho_M^{n+1}(x)\frac{\left(1 + \varepsilon\phi(x)\rho_M^{-n}(x)\right)^{\frac{n+1}{n}} - \left(1 - \varepsilon\phi(x)\rho_M^{-n}(x)\right)^{\frac{n+1}{n}}}{\varepsilon\phi(x)} \\
&= \frac{2(n+1)}{n}\left(\rho_M(x) + \varepsilon^2 R(x)\right),
\end{aligned}$$

where R is the remainder in the Taylor expansion in ε.

Thus,

$$
\begin{aligned}
|K| \langle c(K), e_n \rangle &= \frac{1}{2(n+1)} \int_{S^{n-1}} \left(\frac{\rho_K^{n+1}(x) - \rho_K^{n+1}(-x)}{\varepsilon \phi(x)} \right) x_n \varepsilon \phi(x)\, dx \\
&= \frac{1}{n} \int_{S^{n-1}} \left(\rho_M(x) + \varepsilon^2 R(x) \right) x_n \varepsilon \phi(x)\, dx \\
&= \frac{\varepsilon}{(2\pi)^{n-1}} \int_{S^{n-1}} \left(\rho_M(x) + \varepsilon^2 R(x) \right) \widehat{g}(x)\, dx \\
&= \frac{\varepsilon}{(2\pi)^{n-1}} \int_{S^{n-1}} \left(\widehat{\rho_M}(\xi) + \varepsilon^2 \widehat{R}(\xi) \right) g(\xi)\, d\xi,
\end{aligned}
$$

where we used the spherical Parseval formula.

Idea of proof

Since $g > 0$ is quite arbitrary and $\widehat{\rho_M}$ is sign-changing we can choose g so that

$$\int_{S^{n-1}} \left(\widehat{\rho_M}(\xi) + \varepsilon^2 \widehat{R}(\xi)\right) g(\xi)\, d\xi = 0.$$

Final remark

Remark

The above construction does not work when $n = 3$ and $n = 4$ since there are no non-intersection bodies in these dimensions. We suspect that $\mu(3) = \mu(4) = 1$, but the construction has to be more delicate.

THANK YOU!