The Regge symmetry, confocal conics, and the Schl"afli formula

Arseniy Akopyan (IST Austria)
joint work with
Ivan Izmestiev (University of Fribourg → TU Wien)
Planar magic

\[ s = \frac{a + b + c + d}{2} \]
Planar magic

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The proof
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\[ a \quad b \]

\[ c \quad d \]

\[ s - a \quad s - b \]

\[ s - c \quad s - d \]
The proof
The Ivory theorem
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The Regge symmetry

\[ s = \frac{a + b + c + d}{2} \]
Theorem (G. Ponzano and T. Regge, 1968)

Tetrahedra $\Delta$ and $\bar{\Delta}$ have equal volume.
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Proof.

$$\text{Vol}(\Delta)^2 = \frac{1}{288} \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & a^2 & b^2 & y^2 \\ 1 & a^2 & 0 & x^2 & d^2 \\ 1 & b^2 & x^2 & 0 & c^2 \\ 1 & y^2 & d^2 & c^2 & 0 \end{vmatrix}$$
Theorem (G. Ponzano and T. Regge, 1968)

*Tetrahedra* $\Delta$ and $\bar{\Delta}$ have equal volume.

**Proof.**

\[
\text{Vol}(\Delta)^2 = \frac{1}{288} \begin{vmatrix}
0 & 1 & 1 & 1 & 1 \\
1 & 0 & a^2 & b^2 & y^2 \\
1 & a^2 & 0 & x^2 & d^2 \\
1 & b^2 & x^2 & 0 & c^2 \\
1 & y^2 & d^2 & c^2 & 0 \\
\end{vmatrix}
\]

\[
= \frac{1}{288} \begin{vmatrix}
0 & 1 & 1 & 1 & 1 \\
1 & 0 & (s-a)^2 & (s-b)^2 & y^2 \\
1 & (s-a)^2 & 0 & x^2 & (s-d)^2 \\
1 & (s-b)^2 & x^2 & 0 & (s-c)^2 \\
1 & y^2 & (s-d)^2 & (s-c)^2 & 0 \\
\end{vmatrix} = \text{Vol}(\bar{\Delta})^2
\]
Theorem

Tetrahedra $\Delta$ and $\tilde{\Delta}$ have equal volume in Euclidean, Hyperbolic, and Spherical spaces.

For hyperbolic case it was proved by Y. Mohanty (2003). She also proved scissors congruence of $\Delta$ and $\tilde{\Delta}$. 
More magic!

Theorem

1) The dihedral angles at the x-edge in $\Delta$ and $\bar{\Delta}$ are equal. The same holds for the dihedral angles at the y-edge.
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2) If $\alpha$, $\beta$, $\gamma$, $\delta$ are the dihedral angles at the edges $a$, $b$, $c$, $d$ of $\Delta$, then the dihedral angles at the edges $s - a$, $s - b$, $s - c$, $s - d$ in $\bar{\Delta}$ are equal to $\sigma - \alpha$, $\sigma - \beta$, $\sigma - \gamma$, $\sigma - \delta$, where $\sigma = \frac{\alpha + \beta + \gamma + \delta}{2}$.
Corollary

In three dimensional sphere the Regge symmetry operation and dual operation commute.
Theorem

1) *The dihedral angles at the x-edge in $\Delta$ and $\bar{\Delta}$ are equal. The same holds for the dihedral angles at the y-edge.*
Theorem
2*) The solid angle at the \((x, a, d)\) vertex is equal to the solid angle at the \((x, s - b, s - c)\) vertex etc.;

Area of spherical triangle: \(\alpha + \phi + \delta - \pi\).
For a triangle with fixed angle $\phi$:

$$\text{Area} \propto \tan \frac{z}{2} \tan \frac{t}{2}$$
Unbelievable formula!

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The actual formula is:

$$\tan \frac{\text{Area}}{2} = \frac{\tan \frac{z}{2} \tan \frac{t}{2} \sin \phi}{1 + \tan \frac{z}{2} \tan \frac{t}{2} \cos \phi}$$
For a triangle with fixed angle $\phi$:

$$\alpha + \delta \simeq \tan \frac{z}{2} \tan \frac{t}{2}$$
Unbelievable formula!

For a triangle with fixed angle $\phi$:

\[
\alpha + \delta \approx \tan \frac{z}{2} \tan \frac{t}{2}
\]

\[
\alpha - \delta \approx \frac{\tan \frac{z}{2}}{\tan \frac{t}{2}}
\]
For a triangle with fixed side \( x \):

\[
a + d \approx \tan \frac{\alpha}{2} \tan \frac{\delta}{2}
\]

\[
a - d \approx \frac{\tan \frac{\alpha}{2}}{\tan \frac{\delta}{2}}
\]
Dual version
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\[ \begin{align*}
  a & \quad b \\
  d & \quad c \\
  x & \quad s - d \\
  & \quad x \\
  & \quad s - a \\
  & \quad s - b \\
  & \quad s - c
\end{align*} \]
Dual version

\[ a \quad b \quad d \quad c \]

\[ x \quad s - d \quad s - c \quad s - a \quad s - b \]

\[ a \quad b \quad c \quad d \]

\[ x \]
Dual version

\[s - d \quad x \quad s - c\]
Euqual products
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Theorem (Schlafli)

For every smooth deformation of a spherical or hyperbolic tetrahedron one has
\[ \frac{1}{6} \sum_{i} \ell_i d \theta_i, \]
where the sum is taken over the edges of the tetrahedron, \( \ell_i \) is the length of the i-th side, and \( \theta_i \) is the dihedral angle at the i-th side.

The sign on the right hand side is \( + \) in the spherical and \( - \) in the hyperbolic case.

For a deformation of a Euclidean tetrahedron one has \( \frac{1}{6} \sum_{i} \ell_i d \theta_i = 0 \), but we do not need this formula.
Volume in the Spherical and Hyperbolic cases

Theorem (Schläfli)

For every smooth deformation of a spherical or hyperbolic tetrahedron one has

\[ d \text{Vol} = \pm \frac{1}{2} \sum_{i=1}^{6} \ell_i d\theta_i, \]

where the sum is taken over the edges of the tetrahedron, \( \ell_i \) is the length of the \( i \)-th side, and \( \theta_i \) is the dihedral angle at the \( i \)-th side. The sign on the right hand side is \( + \) in the spherical and \( - \) in the hyperbolic case.
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For a deformation of a Euclidean tetrahedron one has \( \sum_{i=1}^{6} \ell_i d\theta_i = 0 \), but we do not need this formula.
Proof for the Spherical and Hyperbolic cases

By the Schläfli formula one has

\[
\frac{d}{dt} \text{Vol}(\Delta_t) = \pm \frac{1}{2} (a\dot{\alpha} + b\dot{\beta} + c\dot{\gamma} + d\dot{\delta} + x\dot{\phi} + t\dot{\psi})
\]

\[
\frac{d}{dt} \text{Vol}(\bar{\Delta}_t) = \pm \frac{1}{2} \left( (s - a)(\dot{\sigma} - \dot{\alpha}) + \cdots + (s - d)(\dot{\sigma} - \dot{\delta}) + x\dot{\phi} + t\dot{\psi} \right)
\]
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\]

A simple computation

\[
(s - a)(\dot{\sigma} - \dot{\alpha}) + \cdots + (s - d)(\dot{\sigma} - \dot{\delta})
\]

\[
= 4s\dot{\sigma} - (a + b + c + d)\dot{\sigma} - s(\dot{\alpha} + \dot{\beta} + \dot{\gamma} + \dot{\delta}) + a\dot{\alpha} + b\dot{\beta} + c\dot{\gamma} + d\dot{\delta}
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= a\dot{\alpha} + b\dot{\beta} + c\dot{\gamma} + d\dot{\delta}
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\]

\[
= a\ddot{\alpha} + b\ddot{\beta} + c\ddot{\gamma} + d\ddot{\delta}
\]

Therefore for all t

\[
\frac{d}{dt} \text{Vol}(\Delta_t) = \frac{d}{dt} \text{Vol}(\bar{\Delta}_t) \implies \text{Vol}(\Delta) = \text{Vol}(\bar{\Delta}).
\]
Thank you!