On the shadows of a cube

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Prince Rupert’s (1619-1682) problem: Show that a cube can be passed through a hole made in another cube of the same size without splitting the cube into two pieces.

Peter Nieuwland’ extension (1780) : Find the largest cube which can be passed through a unit cube.
\[ a = \sqrt{2}(\sqrt{3} - 1) = 1.03 \]
The square of edge-length $1.06$ is the largest square inside of the cube.

$$\frac{3\sqrt{2}}{4} = 1.06$$
Our question:

Is it possible to make a hole in every possible direction not parallel to a face so that an identical cube can be passed through?

Affirmative answer (A.B., Antal Joós) 2019

Alternative proofs and further similar theorems (A.B, Antal Joós, Mihály Hujter), 2019
Issue of 'translation':

- If something can be 'passed through', can be passed through also by translation?

Claim:
In the plane if a convex disc can be moved through a door, then it can be moved through by translation.
A warning example:

Analogous problem in 3D:
This time the 'door' is a convex hole on the plane. We want to manoeuvre through the door a given convex body.

Rotation can be needed:
The body is a twisted prism with equilateral triangular base. The hole is shown on the right.

Top view of a twisted prizm.

The 'door' on the plane.
Issue of why is Rupert’s problem equivalent to finding the largest inscribed square in a cube.

1. INTRODUCTION. More than three hundred years ago, according to the contemporaneous John Wallis [11, pp. 470–471], Prince Rupert (1619–1682) won a wager that a hole can be cut in one of two equal cubes large enough to permit the second cube to pass through.

Nearly a century later the Dutch scientist Pieter Nieuwland (1764–1794) showed that the largest cube that can be so passed through a cube of side one has side $3\sqrt{2}/4 \approx 1.061$. An accessible discussion with enlightening anaglyphs appears in Ehrenfeucht [8]. In 1950, D. J. E. Schreck [10] gave an interesting, historically based survey of Prince Rupert’s problem and Nieuwland’s extension. Schreck includes a photograph of a model showing the cube in transit.

Nieuwland’s “passage” problem of finding the largest cube that can pass through a unit cube is equivalent to finding the largest square that fits in the unit cube, because once the largest square is located, the hole through the cube having that largest square as its cross section clearly provides the desired passage. In higher dimensions one might seek the side of the largest $m$-dimensional cube that fits in an $n$-dimensional cube of side one. Not much is known about this question, which apparently was first
Corollary:

If a cube of edge length $e$ can be passed through a unit cube, then the unit cube contains a square of side length $e$. 
Main ingredients of the proof:
1. Every projection of a cube contains a unit square (A.B, A. Joós)
2. Convex discs can cover their shadows, a thm by of Géza Kós and JenőTörőcsik:

Conjecure of J. Pach and A. Zalgaller: Disks can cover their shadows.
Proved by
M. Kovaljov (1984)
G. Kós, J. Törőcsik (1990) : short proof
S: square of sidelength $e$
$S$: square of sidelength $e$
Open problem: is it true that every 3D polyhedron has Rupert’s property.