Perron and Frobenius meet Carathéodory
and their other adventures

Alexandr Polyanskii
Moscow Institute of Physics and Technology

joint with Márton Naszódi

Discrete Geometry Days 2
Perron’s Theorem

The simplest form of Perron’s Theorem, 1907

For a square matrix $A$ with positive entries, the spectral radius $\rho(A)$ is an eigenvalue of multiplicity one.
Rankin’s Theorem and its proof

Rankin, 1955

If \( \{v_1, \ldots, v_n\} \) is a set of non-zero unit vectors in \( \mathbb{R}^d \) such that the angle between any two of them is larger than \( \pi/2 \), then \( n \leq d + 1 \).
Rankin’s Theorem and its proof

Rankin, 1955

If \( \{v_1, \ldots, v_n\} \) is a set of non-zero unit vectors in \( \mathbb{R}^d \) such that the angle between any two of them is larger than \( \pi/2 \), then \( n \leq d + 1 \).

Proof, Naszódi+P, 2019+

Suppose \( n = d + 2 \).
Rankin’s Theorem and its proof

Rankin, 1955

If \( \{v_1, \ldots, v_n\} \) is a set of non-zero unit vectors in \( \mathbb{R}^d \) such that the angle between any two of them is larger than \( \pi/2 \), then \( n \leq d + 1 \).

Proof, Naszódi+P, 2019+

Suppose \( n = d + 2 \). Let \( G = \langle v_i, v_j \rangle \) be the Gram matrix.
Rankin’s Theorem and its proof

Rankin, 1955

If \( \{v_1, \ldots, v_n\} \) is a set of non-zero unit vectors in \( \mathbb{R}^d \) such that the angle between any two of them is larger than \( \pi/2 \), then \( n \leq d + 1 \).

Proof, Naszódi+P, 2019+

Suppose \( n = d + 2 \). Let \( G = \langle v_i, v_j \rangle \) be the Gram matrix.

Set \( H = \lambda I_n - G \), where \( \lambda > 1 \)
Rankin’s Theorem and its proof

Rankin, 1955

If \{v_1, \ldots, v_n\} is a set of non-zero unit vectors in \( \mathbb{R}^d \) such that the angle between any two of them is larger than \( \pi/2 \), then \( n \leq d + 1 \).

Proof, Naszódi+P, 2019+

Suppose \( n = d + 2 \). Let \( G = \langle v_i, v_j \rangle \) be the Gram matrix.

Set \( H = \lambda I_n - G \), where \( \lambda > 1 \)

\[ \rightarrow \]

All entries of \( H \) are positive
Rankin’s Theorem and its proof

Rankin, 1955

If \( \{v_1, \ldots, v_n\} \) is a set of non-zero unit vectors in \( \mathbb{R}^d \) such that the angle between any two of them is larger than \( \pi/2 \), then \( n \leq d + 1 \).

Proof, Naszódi+P, 2019+

Suppose \( n = d + 2 \). Let \( G = \langle v_i, v_j \rangle \) be the Gram matrix.

Set \( H = \lambda I_n - G \), where \( \lambda > 1 \)

\[ \downarrow \]

All entries of \( H \) are positive

\[ \downarrow \]

\( \rho(H) \) is the largest eigenvalue of \( H \) of multiplicity one
Proof of Rankin’s Theorem

\( \rho(H) \) is the largest eigenvalue of \( H \) of multiplicity one

\[ G \text{ is the Gram matrix} \]
Proof of Rankin’s Theorem

\( \rho(H) \) is the largest eigenvalue of \( H \) of multiplicity one

\( G \) is the Gram matrix

\[ \Downarrow \]

\[ \alpha_1 \geq \cdots \geq \alpha_d \geq 0 = 0 \] are eigenvalues of \( G \)
$\rho(H)$ is the largest eigenvalue of $H$ of multiplicity one

$G$ is the Gram matrix

\[ \alpha_1 \geq \cdots \geq \alpha_d \geq 0 = 0 \text{ are eigenvalues of } G \]

\[ \lambda = \lambda \geq \lambda - \alpha_d \geq \cdots \geq \lambda - \alpha_1 \text{ are eigenvalues of } H = \lambda I_n - G \]
Proof of Rankin’s Theorem

$\rho(H)$ is the largest eigenvalue of $H$ of multiplicity one

$G$ is the Gram matrix

⇓

$\alpha_1 \geq \cdots \geq \alpha_d \geq 0 = 0$ are eigenvalues of $G$

⇓

$\lambda = \lambda \geq \lambda - \alpha_d \geq \cdots \geq \lambda - \alpha_1$ are eigenvalues of $H = \lambda I_n - G$

⇓

This is a contradiction!
The Perron-Frobenius Theorem

Perron’s Theorem, 1907
For a square matrix with positive entries, the spectral radius is an eigenvalue of multiplicity 1, such that its eigenvector has positive entries.

Frobenius’s Theorem, 1912
For a square matrix with non-negative entries, the spectral radius is an eigenvalue such that one of its eigenvectors has non-negative entries.
Carathéodory’s Theorem, 1907

If \( o \in \mathbb{R}^d \) lies in the convex hull of points \( v_1, \ldots, v_n \in \mathbb{R}^d \), then there is a set \( J \subseteq [n], |J| \leq d + 1 \), such that \( o \in \text{conv}\{v_j : j \in J\} \).
Rankin’s Theorem, 1955

If \( \{v_1, \ldots, v_n\} \) is a set of non-zero vectors in \( \mathbb{R}^d \) such that the angle between any two of them is at least \( \frac{\pi}{2} \), then \( n \leq 2d \).
Frobenius’s Theorem implies . . .

Rankin’s Theorem, 1955

If \( \{v_1, \ldots, v_n\} \) is a set of non-zero vectors in \( \mathbb{R}^d \) such that the angle between any two of them is at least \( \frac{\pi}{2} \), then \( n \leq 2d \).

Steinitz’s Theorem, 1913

If \( o \in \mathbb{R}^d \) is an interior point of the convex hull of points \( v_1, \ldots, v_n \in \mathbb{R}^d \), then there is a set \( J \subseteq [n], |J| \leq 2d \), such that the point \( o \in \text{int conv}\{v_j : j \in J\} \).
Almost-equidistant sets

Definition

A set in $\mathbb{R}^d$ is called *almost-equidistant* if among any three points in the set, some two are at unit distance apart.
Almost-equidistant sets

Definition

A set in $\mathbb{R}^d$ is called *almost-equidistant* if among any three points in the set, some two are at unit distance apart.

Conjecture

An almost-equidistant set in $\mathbb{R}^d$ has $O(d)$ points.
Almost-equidistant sets

Definition
A set in $\mathbb{R}^d$ is called *almost-equidistant* if among any three points in the set, some two are at unit distance apart.

Conjecture
An almost-equidistant set in $\mathbb{R}^d$ has $O(d)$ points.

Theorem, Kupavskii+Mustafa+Swanepoel, P, 2019
An almost-equidistant set in $\mathbb{R}^d$ has $O(d^{4/3})$ points.
Almost-equidistant sets

Definition
A set in $\mathbb{R}^d$ is called *almost-equidistant* if among any three points in the set, some two are at unit distance apart.

Conjecture
An almost-equidistant set in $\mathbb{R}^d$ has $O(d)$ points.

Theorem, Kupavskii+Mustafa+Swanepoel, P, 2019
An almost-equidistant set in $\mathbb{R}^d$ has $O(d^{4/3})$ points.

Idea. Study the eigenvalues of the matrix

$$J_n - \|v_i - v_j\|^2 - I_n,$$

where $J_n$ is the $n$-by-$n$ matrix of ones and $I_n$ is the identity matrix of size $n$. 
The Perron-Frobenius Theorem implies also . . .

Theorem, P, 2019

An almost-equidistant set in $\mathbb{R}^d$ of diameter 1 has at most $2d + 4$ points.
The Perron-Frobenius Theorem implies also . . .

Theorem, P, 2019

An almost-equidistant set in $\mathbb{R}^d$ of diameter 1 has at most $2d + 4$ points.

Question

Prove that the size of an almost-equidistant set of diameter 1 is at most $\lfloor 3(d + 1)/2 \rfloor$. 