Reconstruction in Information Space

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IST Austria
I Bregman Geometry
II Delaunay Mosaics
III Reconstruction
IV Distortion
LEGENDRE TYPE FUNCTIONS

$\Omega \subset \mathbb{R}^d$ open, convex

$F : \Omega \to \mathbb{R}$ is of Legendre type if

(i) $F$ is strictly convex,

(ii) $F$ is differentiable,

(iii) $\| \nabla F(x) \| \to \infty$ as $x \in \partial \Omega$ approaches $\partial \Omega$.

Examples:

$\Omega = \mathbb{R}^d$, $\sigma(x) = \frac{1}{2} \| x \|^2$

$\Omega = \mathbb{R}^d_+$, $E(x) = \sum_i x_i e_i x_i$
Standard Triangle

\[ \Omega = \Delta^2 \subseteq \mathbb{R}^3_+ \]

\[ x \subseteq \Delta^2 \]
\[ E(x) = \sum x_i e^{\lambda x_i} \]
Bregman Divergence

Divergence from $x$ to $y$ w.r.t. $F$ is

$$D_F(x \parallel y) = F(x) - [(x-y)^T \nabla F(y) + F(y)]$$

$$D_F(x \parallel y) = D_F(y \parallel x)$$

$$D_F(x \parallel y) + D_F(y \parallel z) < D_F(x \parallel z).$$
Encoding

Thm. \( |L_x(x)| = -\sum x_i \log x_i \) with frequencies \( x_i \).

\[ |L_y(x)| - |L_x(x)| \] is loss of efficiency.
CONJUGATE

Polarity:

\((a, b) \mapsto y = ax - b\)

Point \( \{ \text{above} \} \) plane \( \iff \) plane* \( \{ \text{above} \} \) point*

\( F^*: \Omega^* \to \mathbb{R} \) s.t. \( y = \nabla F(x) \iff x = \nabla F^*(y) \)

\( D_F(a \| b) = D_{F^*}(\nabla F(b) \| \nabla F(a)) \).
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Balls and Dual Balls

\[ B_F(a; h) = \{ x \in \Omega \mid D_F(a \parallel x) \leq h \} \quad \text{non-convex} \]
\[ B_F^*(a; h) = \{ x \in \Omega \mid D_F(x \parallel a) \leq h \} \quad \text{convex} \]
\[ B_F(a; h) = \nabla F^{-1}(B_F^*(\nabla F(a); h)). \]
non-convex
BREGMAN-VORONOI TESSELLATIONS

\[ \text{dom}_F(a) = \{ x \in \Omega \mid D_F(\text{all} x) \leq D_F(\text{all} y) \ \forall y \in X \} \]

\[ \text{dom}^*_F(a) = \{ x \in \Omega \mid D_F(x, 11a) \leq D_F(y, 11a) \ \forall y \in X \} \]

\[ \text{Vor}_F(X) = \{ \text{dom}_F(a) \} \]

\[ \text{Vor}^*_F(X) = \{ \text{dom}^*_F(a) \} \]

diff. image of such

proj. of convex polyhedron
**Bregman-Delaunay Mosaics**

\[ \text{De}^e_t(X) = \text{Nerve} (\text{Vor}_t(X)) \]

proj. of convex hull

\[ \text{De}^{e*}_t(X) = \text{Nerve} (\text{Vor}^{*}_t(X)) \]

diffeo. image of such
conjugate Shannon geometry
Fisher geometry
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**Drop Function**

\[ g : \text{Def}(X) \to \mathbb{R} \text{ with} \]

\[ g(Q) = \min \{ h \mid \bigcap_{a \in Q} [B(a; h) \cap \text{dom}(a)] \neq \emptyset \} \]

is equivalent to slowly dropping lifted points

\[ \text{Alpha}_h(x) = g^{-1}(-\infty, h] \]
Fisher Metric

\[ \varphi \]

isometry
Weighted Euclidean Case

triangulation acc. to Shannon geometry

drop function acc. to Euclidean geometry
Shannon geometry
conjugate Shannon geometry
Euclidean geometry
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Compare Mosaics

Jaccard distance is

\[ J(D_0, D_1) = \frac{\#(D_0 \cap D_1)}{\#D_0 + \#D_1 - \#(D_0 \cap D_1)} \]

is normalized in \([0,1]\).
conjugate Shannon geometry
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COMPARE DROP FUNCTIONS

Special case: \( g_0, g_1 : D \rightarrow \mathbb{R} \)
then count inversions.

\[ \#\text{Inv} (Sh, \omega E_u) = 0.466. \]

In general: \( W_\infty (Dgm(g_0), Dgm(g_1)) \).
conjugate Shannon geometry
Fisher geometry

Delaunay
weighted Euclidean geometry
THANK YOU