Equipartitions and Mahler’s conjecture

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Discrete Geometry Days II

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Budapest
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$$P(K) \leq P(B_{\infty}^n).$$  

In 1939, Mahler conjectured (and proved for $n = 2$) that

$$P(K) \geq P(B_{\infty}^n).$$
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I want to talk about the equipartition result. If $K$ minimises the Mahler volume, then the partition, in some sense, tells you why $K$ is similar to $B_3^\infty$. 
An equipartition result

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By symmetry, we only need $3 + 1 + 1 + 1$ equalities.
Configuration Space
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There is a group acting here!
Test Map

\[ H_2 \quad H_3 \]
$f(u, v) = (\begin{array}{ccc}
- & - \\
\end{array}, \begin{array}{ccc}
- & - \\
\end{array}, \begin{array}{ccc}
- & - \\
\end{array})$
We want zeros of $f$. 

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\begin{align*}
\mathbf{f}(u, v) &= (\begin{bmatrix} -x \\ -y \\ -z \end{bmatrix}, \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \begin{bmatrix} -x \\ -y \\ -z \end{bmatrix})
\end{align*}
\]
A special polynomial

Let $f_0(u, v, w) = (u_x, v_y, w_y, w_x + v_x, w_x - v_x)$. 
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\begin{align*}
Z(f_0) & \quad \bullet \quad Z(f) \\
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About the rest...

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\[ = \sum \bigcirc \bigcirc \bigcirc \bigcirc \geq \sum \text{planar case} = \frac{32}{3} \cdot \square \]
¡Thank you!