Random parallelotopes in isotropic vector systems

Ferenc Fodor

Department of Geometry
Bolyai Institute
University of Szeged

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This talk is based on the joint paper:

the Euclidean scalar product $\langle \cdot , \cdot \rangle$ and
the induced norm $| \cdot |$
the unit ball $B^d$ of $\mathbb{R}^d$ centred at the origin and
its boundary $\partial B^d = S^{d-1}$ the unit sphere
the $d$-dimensional identity map $\text{Id}_d$
convex bodies
the tensor product:

For $u, v \in \mathbb{R}^d$ let $u \otimes v : \mathbb{R}^d \to \mathbb{R}^d$ denote the tensor product, that is, $(u \otimes v)(x) = \langle v, x \rangle u$ for every $x \in \mathbb{R}^d$. The matrix of $u \otimes v$ is $uv^T$, where the vectors are column vectors.

Remark

For $u \in S^{d-1}$, $u \otimes u$ is the orthogonal projection to the linear subspace spanned by $u$, that is, onto the line of $u$. 
**Theorem (John, Ball)**

Let $K$ be a $d$-dimensional convex body. Then there exists a unique maximal volume ellipsoid contained in $K$. Moreover, this maximal volume ellipsoid is the unit ball if and only if there exists vectors $u_1, \ldots, u_m \in \partial K \cap S^{d-1}$ and (positive) $c_1, \ldots, c_m$ such that

\[ \sum_{i=1}^{m} c_i u_i \otimes u_i = \text{Id}_d, \quad \text{and} \]
\[ \sum_{i=1}^{m} c_i u_i = 0. \]

- (1) is called the (discrete) isotropy condition.
- (2) is the centeredness condition.
- If some unit vectors $u_1, \ldots, u_m \in S^{d-1}$ satisfy (1) and (2), then they form a John decomposition of the identity.
Dvoretzky–Rogers lemma

**Lemma (Dvoretzky, Rogers)**

Let $u_1, \ldots, u_m \in S^{d-1}$ and $c_1, \ldots, c_m > 0$ such that

$$\sum_{i=1}^{m} c_i u_i \otimes u_i = I_d.$$

Then there exists an orthonormal basis $b_1, \ldots, b_d$ of $\mathbb{R}^d$ and a subset $\{x_1, \ldots, x_d\} \subset \{u_1, \ldots, u_m\}$ with $x_j \in \text{lin}\{b_1, \ldots, b_j\}$ and

$$\sqrt{\frac{d - j - 1}{d}} \leq \langle x_j, b_j \rangle \leq 1.$$

Let $P$ be the parallelotope spanned by $x_1, \ldots, x_d$. The volume of $P$ is bounded from below by

$$V(P) = |\det(x_1, \ldots, x_d)| \geq \frac{\sqrt{d!}}{d^{\frac{d}{2}}}.$$
The number of vectors in a John decomposition

Theorem (John, Ball, Pelczynski, Gruber and Schuster)

If a set of unit vectors satisfies (1) (resp., (1) and (2)) with some positive scalars $c_i'$, then a subset of $m$ elements also satisfies (1) (resp., (1) and (2)) with some positive scalars $c_i$, where

$$d + 1 \leq m \leq d(d + 1)/2$$

(resp., $d + 1 \leq m \leq d(d + 3)/2$).

We give an alternate proof of this Theorem for two reasons:

- We use the part when only (1) is assumed, which is only implicitly present in the original paper.
- The original result is described in the context of John’s theorem. We give a presentation where the linear algebraic fact and its use in convex geometry are separated.
Proof

- (1) holds with some positive scalars \(c_i\), if and only if, the matrix \(I_d d/d\) is in the convex hull of the set \(\mathcal{A} = \{v_i \otimes v_i : i = 1, \ldots, m\}\) in the real vector space of \(d \times d\) matrices.

- The set \(\mathcal{A}\) is contained in the subspace of symmetric matrices with trace 1, which is of dimension \(d(d + 1)/2 - 1\). Carathéodory’s theorem now yields the desired upper bound.

- If both (1) and (2) are assumed, then we lift the vectors into \(\mathbb{R}^{d+1}\) as follows. Let \(\hat{v}_i = \sqrt{\frac{d}{d+1}} (v_i, 1/\sqrt{d}) \in \mathbb{R}^{d+1}\).

- Then \(|\hat{v}_i| = 1\), and (1) holds for the vectors \(\hat{v}_i\) with some positive scalars \(\hat{c}_i\) if and only if (1) and (2) hold for the vectors \(v_i\) with scalars \(c_i = \frac{d}{d+1} \hat{c}_i\).

- Now, \(\hat{v}_i \otimes \hat{v}_i, i = 1, \ldots, m\) are symmetric \((d + 1) \times (d + 1)\) matrices of trace one, and their \((d + 1, d + 1)\)th entry is \(1/(d + 1)\). The dimension of this subspace of \(\mathbb{R}^{(d+1)\times(d+1)}\) is \(d(d + 3)/2 - 1\).
Definition (Isotropic measure)

We say $\mu$ is an isotropic measure if it is a probability measure in $\mathbb{R}^d$ which satisfies the following:

$$\int_{\mathbb{R}^d} x \otimes x \, d\mu(x) = \text{Id}_d,$$

and

$$\int_{\mathbb{R}^d} x \, d\mu(x) = 0.$$

Note that taking the trace in the isotropy condition (1) yields $\sum_{i=1}^m c_i = d$. Thus, the Borel measure $\mu_K$ on $\sqrt{d} S^{d-1}$ with

$\text{supp} \, \mu_K = \{\sqrt{d} u_1, \ldots, \sqrt{d} u_m\}$ and $\mu_K(\{\sqrt{d} u_i\}) = c_i / d$

($i = 1, \ldots, m$) is a discrete isotropic measure.
Pivovarov (2010) proved the following statement about the volume of random parallelotopes spanned by \(d\) independent, isotropic vectors.

**Lemma (Pivovarov, 2010)**

Let \(x_i, i = 1, \ldots, d\) be a random vectors according to the isotropic measure \(\mu_i\) in \(\mathbb{R}^d\). Assume that \(x_1, \ldots, x_d\) are linearly independent with probability 1. Then

\[
\mathbb{E}([\det(x_1, \ldots, x_d)]^2) = d!
\]
We extend Pivovarov’s lemma to a more general class of measures in the following way.

**Lemma (F., Naszódi, Zarnócz (2018))**

Let $x_1, \ldots, x_d$ be independent random vectors distributed according to the measures $\mu_1, \ldots, \mu_d$ satisfying the isotropy condition. Assume that $\mu_i(\{0\}) = 0$ for $i = 1, \ldots, d$. Then

$$\mathbb{E}([\det(x_1, \ldots, x_d)]^2) = d!$$

We note that Lutwak, Yand and Zhang (2004) established similar results for the special case of discrete isotropic measures, which could also be used to prove the volumetric bounds in the following Theorem.

The meaning of this extended lemma is that, in fact, the volume bound in the DV lemma can be considered as an expectation.
We give a probabilistic proof of the following theorem.

**Theorem (Pelczynski, Szarek (1989), González, Schymura (2018), F., Naszódi, Zarnócz (2018))**

Let \( u_1, \ldots, u_m \in S^{d-1} \) be unit vectors satisfying the discrete isotropy condition with positive weights \( c_1, \ldots, c_m \). Then there is a subset \( \{ x_1, \ldots, x_d \} \subset \{ u_1, \ldots, u_m \} \) with

\[
[\det(x_1, \ldots, x_d)]^2 \geq \gamma(d, m) \frac{d!}{d^d},
\]

where \( m = \min\{ m, d(d + 1)/2 \} \) and \( \gamma(d, m) = \frac{m}{d!} \left( \frac{m}{d} \right)^{-1} \)

\( \gamma(d, m) \) is the (square of the) scalar of improvement compared to the result of the Dvoretzky–Rogers lemma. In certain cases the improvement is exponentially better, in others it is even optimal.
### Theorem (Pelczynski, Szarek (1989), González, Schymura (2018), F., Naszódi, Zarnócz (2018))

1. $\gamma(d, \overline{m}) \geq \gamma(d, (d + 1)/2) \geq 3/2$ for any $m \geq d \geq 2$. Moreover $\gamma(d, d(d + 1)/2) \nearrow e$ as $d \to \infty$.

2. For a fixed $c > 1 + 1/d$ we consider the case when $m \leq cd$. Then

   $$\gamma(d, m) \geq \gamma(d, \lceil cd \rceil) \sim \sqrt{\frac{c-1}{c}} \left( \frac{c-1}{c} \right)^{(c-1)d} e^d,$$

   as $d \to \infty$.

3. For a fixed $k \geq 1$ we consider the case when $m \leq d + k$. Then

   $$\gamma(d, m) \geq \gamma(d, d + k) \sim \frac{k!e^k}{\sqrt{2\pi}} \frac{e^d}{(d + k)^{k+1/2}}, \text{ as } d \to \infty.$$
Idea of the proof

- The main idea of the proof is that we notice that the expectation of the squared volume of random parallelotopes in discrete isotropic measures is the same as the square of the volume bound in the Dvoretzky-Rogers lemma.
- So the DV lemma gives the average, and we want the maximum.
- However, in the discrete case the maximum is strictly larger than the expectation (average) because the volume of the random parallelepiped is 0 with positive probability.
- We prove that the maximum is smallest in the case when the vectors in the measure are equally probable.
- Using the Gruber-Schuster theorem, we may always assume that the isotropic measure has at most $d(d + 1)/2$ elements. With the help of this, we may give an explicit bound for the maximum that only depends on the dimension.
- This bound is (much) better for measures with few vectors.
Let $k = 1$, that is, when $K$ is the regular simplex whose inscribed ball is $B^d$. Then the John decomposition of the identity determined by $K$ consists of $d + 1$ unit vectors that determine the vertices of a regular $d$-simplex $\Delta_d$ inscribed in $B^d$, and note that $\text{Vol}(\Delta_d) = (d + 1)^{d+1}/(d^{d/2}d!)$. Clearly, in this John decomposition of the identity, the volume of the simplex determined by any $d$ of the vectors $u_1, \ldots, u_{d+1}$ is

$$\text{Vol}(\Delta_d)/(d + 1) = \frac{(d + 1)^{d-1}}{d^{d/2}d!}.$$

By the Main Theorem, we obtain that

$$\max[\det(u_{i_1}, \ldots, u_{i_d})]^2 \geq \frac{(d + 1)^{d-1}}{d!} \cdot \frac{d!}{d^d} = \frac{(d + 1)^{d-1}}{d^d},$$

thus, the Main Theorem is sharp in this case.
The following proposition yields a lower bound on the probability that a randomly chosen set of vectors span a bigger parallelotope than the one provided in the Dvoretzky–Rogers lemma.

**Proposition (F., Naszódi, Zarnócz (2019))**

Let \( \lambda \in (0, 1) \). With the earlier notations and assumptions, if we choose the vectors \( x_1, \ldots, x_d \) independently according to the distribution \( \mathbb{P}(x_\ell = u_i) = c_i / d \), then with probability at least \((1 - \lambda)e^{-d}\) we have that

\[
[\det(x_1, \ldots, x_d)]^2 \geq \lambda \gamma(d, \overline{m}) \cdot \frac{d!}{d^d}.
\]
Concluding remarks

Some open questions:

- What isotropic measures satisfy all of the optimality conditions (beside the simplex)?

- Are there finite sets of unit vectors with the following property: the (square of the) volume of the random parallelotopes (spanned by $d$ independent vectors) can take only two values: 0 and another positive number? Beside the John decomposition of the regular simplex, cube...?

- Can one control the minimum while keeping the measure isotropic?

- How about small examples (small dimensions and a small number of vectors)?
Thank you for your attention.