Almost regular triangles

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Triangles $T$ and $T'$ with angles $x, y, \gamma$ and $x', y', \gamma'$ are $\varepsilon$-similar if

$$|x-x'|, |y-y'|, |\gamma-\gamma'| < \varepsilon$$

$\varepsilon$ is small, smaller than $\alpha, \beta, \gamma$...
Def \( h(n, T, \varepsilon) = \) maximal number of \( \varepsilon \)-similar (to \( T \)) triangles in an \( n \)-element planar set

Determine \( h(n, T, \varepsilon) \)

FACT. \( h(n, T, \varepsilon) \geq \frac{n^3 - n}{24} \) for \( n \geq 3 \)
with \( f(n) = h(n, T, r) \)

\[
f(a+b+c) \geq abc + f(a) + f(b) + f(c)
\]

define \( h(n) \), \( n = 0, 1, 2, \ldots \) as the maximal lower bound:

\[
h(n) = \max \{ abc + h(a) + h(b) + h(c) : a + b + c = n \}
\]

\( a, b, c \geq 0 \) in integers \( h(0) = h(1) = h(2) = 0 \)

\( h(3) = 1 \)

Induction shows that

\[
\frac{h^3}{24} - O(h \log h) < h(n) \leq \frac{h^3 - h}{24}
\]

with equality on RHS if \( n = 3^k \).
\[
\begin{align*}
    h(n) &\leq \max \left\{ abc + \frac{a^3-a}{24} + \frac{b^3-b}{24} + \frac{c^3-c}{24} : \right. \\
    &\left. \quad a+b+c = n, \quad a, b, c \geq 0 \right\} \\
    &= \frac{n^3-n}{24} + \max \left\{ \frac{3}{4} \left( abc - \frac{a^3b+ab^3+b^3c+bc^3+c^3a+ac^3}{6} \right) : \right. \\
    &\left. \quad a+b+c = n, \quad a, b, c \geq 0 \right\} \\
    &\leq \frac{n^3-n}{24}. 
\end{align*}
\]
Theorem 1. For the regular triangle $T \exists \varepsilon_0 > 1^\circ$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and for all $n$:

$$h(n, T, \varepsilon) = h(n)$$

When $n$ is a power of 3, then $h(n, T, \varepsilon) = \frac{h^3 - n}{24}$.

Theorem 2. Let $T$ be a triangle whose angles are between $60^\circ - \frac{\varepsilon_0}{2}$ and $60^\circ + \frac{\varepsilon_0}{2}$. If $\varepsilon \in (0, \frac{\varepsilon_0}{2})$, then

$$h(n, T, \varepsilon) = h(n)$$
Proof of Thm 1 is technical:

1. Choose a well-behaved maximizer $P_n^*$

2. $xy \in P_n$ is a diameter of $P_n$

3. $z \in P_n$ and $xyz$ is almost regular

show that

$P_n \subset N(x) \cup N(y) \cup N(z)$

show that

there is no almost regular triangle in $P_n$ with

2 vertices in $N(x)$ and one in $N(y)$ or $N(z)$
Define \( h(T, \varepsilon) = \lim_{n \to \infty} \frac{h(n, T, \varepsilon)}{h^3} \).

Thm 3. This limit exists and is at least \( \frac{1}{24} \).

Further

\[ h(T, \varepsilon) (h^3 - n) \geq h(n, T, \varepsilon) \geq h(T, \varepsilon) \eta(n-1)(n-2) \]

follows from

\[ \frac{h(n, T, \varepsilon)}{\binom{n}{3}} \geq \frac{h(n+1, T, \varepsilon)}{\binom{n+1}{3}} \]
Is \( h(T, \varepsilon) = \frac{1}{2^4} \) always?

**Constructions.** \( Q = \{ q_1, q_2, \ldots, q_3 \} \subset \mathbb{R}^2 \), \( T \) a triangle \( F(Q, T, \varepsilon) \) 3-uniform hypergraph with vertex set \( \{ v_1, \ldots, v_3 \} \) and \( ijk \) an edge of \( F \) is \( q_i, q_j, q_k \) is \( \varepsilon \)-similar to \( T \). \( P_i \subset \mathbb{R}^2 \), \( |P_i| = y_i \) (4)

Then \( \exists s = s(Q, T, \varepsilon) > 0 \) such that:
\( \mathcal{F} = \{ \text{four blue triangles} \} \) 

\( D_1, \ldots, D_r \) disks of radius \( g \)

Then every triangle \( \triangle p_ip_jp_k \) with \( p_i \in D_i, p_j \in D_j, p_k \in D_k \) is similar to \( T \) if \( ijk \in \mathcal{F} \) but all other triangles are not except possibly when \( i = j = k \).

\( h = 1 |P| = \sum y_i \). 

Put a homothetic copy of \( P_i \) into \( D_i \) to get \( P \subset \mathbb{R}^2 \).
\[ h(P, T, \varepsilon) = \sum_{i=1}^{r} h(P_i, T, \varepsilon) + \sum_{ijk \in \mathcal{F}} y_i y_j y_k \]

\hspace{1cm} p(y_1, \ldots, y_r) \text{ polynomial} \]

Using \((T, \varepsilon)\)-optimal \(P_i: h(P_i, T, \varepsilon) = h(y_i, T, \varepsilon)\) gives

\[ h(h, T, \varepsilon) \geq \sum_{i} h(y_i, T, \varepsilon) + p(y_1, \ldots, y_r) \]

We have \(n \) fixed (and \( T, \varepsilon, \mathcal{G} \)) how to choose \( y_i \)?
Define
\[ y_i = n \sum x_i \alpha (n x_i) \quad \text{s.t.} \quad \sum y_i = n \]

Given \( P_n \) (not unique) but

**Lemma** \( \forall T \exists \varepsilon(T) > 0 \quad \forall \varepsilon \in (0, \varepsilon(T)) \)

\[
\left| h(P_n, T \varepsilon) - n^2 \frac{p(x_1, \ldots, x_r)}{1 - (x_1^3 + \ldots + x_r^3)} \right| \leq \frac{r}{1 - \max x_i} \cdot n^2
\]

**Remark** \( x_i > 0 \) \( (\forall i) \) \( \Rightarrow \) \( \max x_i < 1 \).
We want to minimize

\[
p(x_1, \ldots, x_r) \frac{1 - (x_1^3 + \ldots + x_r^3)}{1 - (x_1 + \ldots + x_r)}
\]

subject to \( x_1 + \ldots + x_r = 1 \), \( x_i > 0 \) (for i)
Example 0: \( r = 3 \) \( \mathcal{F} = \{1, 2, 3\} \)

\[
x_i = \frac{1}{3}
\]

gives

\[
h(n, T, \varepsilon) \geq \frac{n^3}{2^4} + O(n^2)
\]

Example 1. \( r = 4 \)

\( \mathcal{F} = \text{all 4 triples} \)

\[
x_i = \frac{1}{4}
\]

\[
h(n, T, \varepsilon) \geq \frac{n^3}{1^5} + O(n^2)
\]
Example 3. \( r = 5 \) \( F = \{8 \text{ triples}\} \)

\[
x_1 = x_2 = x_3 = x_4 = x, \quad x_5 = 1 - 4x
\]

\[
f(x) = \frac{x - 3x^2}{3(1 - 4x + 5x^2)} - \epsilon(x) \in (0, \frac{1}{4})
\]

\[
\max \text{ at } x = \frac{3 - \sqrt{2}}{7} \quad \text{when} \quad f(x) = \frac{1}{6 \sqrt{2} + 6} = \frac{1}{14.4852...}
\]

\[
h_n(n, t, \epsilon) \geq \frac{n^3}{14.4852...} + O(n^2)
\]

overall max?
Example 4. \( r = 4 \) \( F = \{3 \text{ triples}\} \)

\( x_1 = x_2 = x_3 = x_4 = 1 - 3x \), \( x \in (0, \frac{1}{3}) \)

\( h(n, T, \varepsilon) \geq \frac{h^3}{18.797...} + O(n^2) \)

Example 5. \( r = 4 \) \( F = \{124, 239, 341\}\)

\( \alpha = 40.2^\circ \), \( T = (\alpha, 2\alpha, 180^\circ - 3\alpha) \)

\( (\min 3\alpha)^3 = \min \alpha (\min 2\alpha)^2 \)

\( p(x_1, x_2, x_3, x_4) \) same as in Example 4.

So \( h(n, T, \varepsilon) \) is the same.
Example 6. \( r = 6 \), \( T = (30^\circ, 60^\circ, 90^\circ) \)

\[ F = \{ 12 \text{ triples} \} \quad x_i = \frac{1}{6} \]

\[ h(n, T, \varepsilon) \geq \frac{n^3}{17.5} + O(n^2) \]

Example 7. \( r = 5 \) \( T = (36^\circ, 36^\circ, 108^\circ) \)

\[ F = \{ 35 \text{ triples} \} \quad x_i = \frac{1}{5} \]

\[ h(n, T, \varepsilon) \geq \frac{n^3}{24} + O(n^2) \]
Example 8. Same as No previous Example just \( T - (36^\circ, 72^\circ, 72^\circ) \)

In both cases, \( \varphi(p_n, T, \varepsilon) = \frac{n^3 - n}{24} \)

if \( n \) is a power of 5.

Example 9. \( v = 7 \), \( G \) regular \( 7 \)-gon

\( F = \{ 14 \text{ triples} \} \) ... \( \varphi(p_n, T, \varepsilon) \geq \frac{n^3}{24} - O(n^2) \)

and \( \varphi(p_n, T, \varepsilon) = \frac{n^3 - n}{24} \) if \( n \) is a power of 7.
Example 10. \( r = 5 \), \( \mathcal{F} = \{ 01z, 01 \frac{1}{1-z}, 01 \frac{2-z}{z}, z \} \) \( \frac{1}{1-z} \frac{2-z}{z} \)

\[ x = \left( \frac{1}{5}, \frac{1}{5}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9} \right) \]

\[ h(h, T, \varepsilon) \geq \frac{h^3}{24} + O(h^2) \]

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Fact \( z = \frac{1}{1-z} \frac{2-z}{z} \) is similar to \( 01z \)
Is $h(T(\varepsilon)) = \frac{1}{24}$ almost always?

Space of triangle shapes:

$(x, y, z) \in \mathbb{R}^3 \quad x + y + z = 1$

$x, y, z > 0$
There are few examples with $h(T_{12}) > \frac{1}{24}$:

\[ \frac{1}{14.48} = \frac{1}{24} \]

\[ \frac{1}{18.78} \]

\[ \frac{1}{3.61} \]

\[ \frac{1}{15} \]

\[ \frac{2\pi}{3} \]

\[ \frac{\pi}{6} \]

\[ \frac{\pi}{4} \]

\[ \frac{\pi}{3} \]

\[ \frac{\pi}{2} \]

\[ \alpha = \frac{\pi}{2} \]
Thm 4. For almost every triangle $T$

$\exists \epsilon = \epsilon(T) > 0$ such that

$h(u, T, \varepsilon) \leq 0.25108 \left( \frac{h}{3} \right) (1 + o(1))$

$\approx \left( \frac{1}{24} + 0.00018 \right) h^3$

Can improve to 0.25072
Proof uses Turán's theory of extremal hypergraphs and computers (flag algebra computations).

$L$ - a finite family of 3-uniform hypergraphs

"forbidden hypergraphs"

Determine the maximal number of edges of a 3-uniform hypergraph $H$ on $n$ vertices if it does not contain any member of $L$, $\text{ex}(n, L)$
\[ \mathcal{K}_4^{-} = \{124, 134, 234\} \quad \mathcal{C}_5 = \{123, 234, 345, 451, 512\} \]

\[ \mathcal{L} = \{ \mathcal{K}_4^{-}, \mathcal{C}_5 \} \]

Thus (Falgas-Ravry and Vaughan, 2013)

\[ (0.25 + o(1))(\binom{n}{3}) \leq \text{ex}(n, \mathcal{L}) \leq 0.25108(\binom{n}{3}) \]

Conjecture: \[ \text{ex}(n, \mathcal{L}) = (\frac{1}{4} + o(1))(\binom{n}{3}) \]
Given a triangle $T$ with angles $\alpha, \beta, \gamma$ (in radians) an equation

$$n_1 \alpha + n_2 \beta + n_3 \gamma + n_4 \pi = 0$$

is a non-trivial equation for $T$ if $(n_1, n_2, n_3, n_4)$ is linearly independent of $(1, 1, 1, -1)$ and every $n_i$ is an integer with $|n_i| \leq 5$.

Note that \( \alpha + \beta + \gamma - \pi = 0 \) for every triangle.
Given $Q \subset \mathbb{R}^2$, $|Q|=r$ and $T$, define $F(Q,T)$ as the $3$-uniform hypergraph on vertex set $Q$ with $x,y,z \in Q$ forming an edge if triangle $xyz$ is similar to $T$. 
Lemma 1. $|Q|=4$ and $F(Q,T)$ contains a copy of $K_4$. Then the angles of $T$ satisfy a non-trivial equation.

Lemma 2. $|Q|=5$ and $F(Q,T)$ contains a copy of $C_5$. Then the angles of $T$ satisfy a non-trivial equation.
Thanks!