Uniform contractions of balls revisited

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The Kneser-Poulsen conjecture (1954-1955):

We denote the Euclidean norm of a vector $\mathbf{p}$ in the $d$-dimensional Euclidean space $\mathbb{E}^d$ by $|\mathbf{p}| := \sqrt{\langle \mathbf{p}, \mathbf{p} \rangle}$, where $\langle \cdot, \cdot \rangle$ is the standard inner product. The $d$-dimensional volume (i.e., Lebesgue measure in $\mathbb{E}^d$) of a compact set $A \subset \mathbb{E}^d$ is denoted by $V_d(A)$. In this paper, for simplicity $V_d(\emptyset) = 0$. The closed Euclidean ball of radius $r$ centered at the point $\mathbf{p} \in \mathbb{E}^d$ is denoted by $B^d[\mathbf{p}, r] := \{ \mathbf{q} \in \mathbb{E}^d \mid |\mathbf{p} - \mathbf{q}| \leq r \}$. We say that the (labeled) point set $Q := \{ \mathbf{q}_1, \ldots, \mathbf{q}_N \} \subset \mathbb{E}^d$ is a contraction of the (labeled) point set $P := \{ \mathbf{p}_1, \ldots, \mathbf{p}_N \} \subset \mathbb{E}^d$ in $\mathbb{E}^d$, $d > 1$ if $|\mathbf{q}_i - \mathbf{q}_j| \leq |\mathbf{p}_i - \mathbf{p}_j|$ holds for all $1 \leq i < j \leq N$. In 1955, M. Kneser [13] and E. T. Poulsen [15] independently stated the conjecture that if $Q = \{ \mathbf{q}_1, \ldots, \mathbf{q}_N \}$ is a contraction of $P = \{ \mathbf{p}_1, \ldots, \mathbf{p}_N \}$ in $\mathbb{E}^d$, $d > 1$, then

$$V_d \left( \bigcup_{i=1}^{N} B^d[\mathbf{p}_i, r] \right) \geq V_d \left( \bigcup_{i=1}^{N} B^d[\mathbf{q}_i, r] \right)$$

(1)

holds for all $N > 1$ and $r > 0$.


If $Q = \{q_1, \ldots, q_N\}$ is a contraction of $P = \{p_1, \ldots, p_N\}$ in $\mathbb{E}^d$, $d > 1$, then

$$V_d \left( \bigcap_{i=1}^{N} B^d[p_i, r] \right) \leq V_d \left( \bigcap_{i=1}^{N} B^d[q_i, r] \right)$$

holds for all $N > 1$ and $r > 0$.


in 1987, M. Gromov [9] published a proof of (2) for all $N \leq d + 1$ and for not necessarily congruent balls and conjectured that his result extends to spherical $d$-space $S^d$ (resp., hyperbolic $d$-space $\mathbb{H}^d$) for all $d > 1$.


in 1991 V. Klee and S. Wagon [12] asked whether (2) holds for not necessarily congruent balls as well. (We note that in [12] the *Kneser-Poulsen conjecture* under (1) is stated in its most general form, that is, for not necessarily congruent balls.)

On the status of (1) and (2) in short:

The author and R. Connelly [2] proved (1) as well as (2) for not necessarily congruent circular disks and for all \( N > 1 \) in \( \mathbb{E}^2 \).


However, both (1) and (2) remain open in \( \mathbb{E}^d \) for all \( d \geq 3 \).

For a number of partial results proved on (1) and (2) see:

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Uniform contractions have been introduced for the study of (1) and (2) as follows:

Definition

Let $Q := \{q_1, \ldots, q_N\} \subset \mathbb{E}^d$ be a uniform contraction of the (labeled) point set $P := \{p_1, \ldots, p_N\} \subset \mathbb{E}^d$ with separating value $\lambda > 0$ in $\mathbb{E}^d$, $d > 1$ if $|q_i - q_j| \leq \lambda \leq |p_i - p_j|$ holds for all $1 \leq i < j \leq N$. The problem of proving (1) as well as (2) for uniform contractions has been investigated in the recent papers [3], [4], and [5].

References


The Kneser–Poulsen Conjecture for Special Contractions

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**Theorem 1.4** Let $d, N \in \mathbb{Z}^+$, $k \in [d]$ and let $q \in \mathbb{E}^{d \times N}$ be a uniform contraction of $p \in \mathbb{E}^{d \times N}$ with some separating value $\lambda \in (0, 2]$. If $N \geq (1 + \sqrt{2})^d$ then

$$V_k\left( \bigcap_{i=1}^{N} B[p_i] \right) \leq V_k\left( \bigcap_{i=1}^{N} B[q_i] \right). \quad (1)$$

**Theorem 1.5** Let $d, N \in \mathbb{Z}^+$, and let $q \in \mathbb{E}^{d \times N}$ be a uniform contraction of $p \in \mathbb{E}^{d \times N}$ with some separating value $\lambda \in (0, 2]$. If $N \geq c^d d^{2.5d}$ then

$$V_d\left( \bigcup_{i=1}^{N} B[p_i] \right) \geq V_d\left( \bigcup_{i=1}^{N} B[q_i] \right), \quad (2)$$

where $c > 0$ is a universal constant.
From $r$-dual sets to uniform contractions

KÁROLY BEZDEK

Definition 1. For a set $X \subseteq \mathbb{M}^d$, $d > 1$ and $r \in \mathbb{R}_+$ let the $r$-dual set $X^r$ of $X$ be defined by $X^r := \bigcap_{x \in X} B_{\mathbb{M}^d}[x, r]$. If the interior $\text{int}(X^r) \neq \emptyset$, then we call $X^r$ the $r$-dual body of $X$.

Theorem 1. Let $A \subseteq \mathbb{M}^d$, $d > 1$ be a compact set of volume $V_{\mathbb{M}^d}(A) > 0$ and $r \in \mathbb{R}_+$. If $B \subseteq \mathbb{M}^d$ is a ball with $V_{\mathbb{M}^d}(A) = V_{\mathbb{M}^d}(B)$, then $V_{\mathbb{M}^d}(A^r) \leq V_{\mathbb{M}^d}(B^r)$.

Note that the Gao–Hug–Schneider theorem is a special case of Theorem 1 namely, when $\mathbb{M}^d = S^d$ and $r = \frac{\pi}{2}$. As this theorem of [10] is often called a spherical counterpart of the Blaschke–Santaló inequality, one may refer to Theorem 1 as a Blaschke–Santaló-type inequality for $r$-duality in $\mathbb{M}^d$.

Theorem 2. (i) Let $d \in \mathbb{Z}$ and $\delta, \lambda \in \mathbb{R}$ be given such that $d > 1$ and $0 < \lambda \leq \sqrt{2}\delta$. If $Q := \{q_1, \ldots, q_N\} \subset \mathbb{E}^d$ is a uniform contraction of $P := \{p_1, \ldots, p_N\} \subset \mathbb{E}^d$ with separating value $\lambda$ in $\mathbb{E}^d$ and $N \geq (1 + \sqrt{2})^d$, then $V_{\mathbb{E}^d}(P^\delta) < V_{\mathbb{E}^d}(Q^\delta)$.

(ii) Let $d \in \mathbb{Z}$ and $\delta, \lambda \in \mathbb{R}$ be given such that $d > 1, 0 < \delta < \frac{\pi}{2}$, and $0 < \lambda < \min \left\{ \frac{2\sqrt{2}}{\pi}\delta, \pi - 2\delta \right\}$. If $Q := \{q_1, \ldots, q_N\} \subset \mathbb{S}^d$ is a uniform contraction of $P := \{p_1, \ldots, p_N\} \subset \mathbb{S}^d$ with separating value $\lambda$ in $\mathbb{S}^d$ and $N \geq 2ed\pi^{d-1} \left( \frac{1}{2} + \frac{\pi}{2\sqrt{2}} \right)^d$, then $V_{\mathbb{S}^d}(P^\delta) < V_{\mathbb{S}^d}(Q^\delta)$.

(iii) Let $d, k \in \mathbb{Z}$ and $\delta, \lambda \in \mathbb{R}$ be given such that $d > 1, k > 0$ and $0 < \frac{\sinh k}{\sqrt{2k}} \lambda \leq \delta < k$. If $Q := \{q_1, \ldots, q_N\} \subset \mathbb{H}^d$ is a uniform contraction of $P := \{p_1, \ldots, p_N\} \subset \mathbb{H}^d$ with separating value $\lambda$ in $\mathbb{H}^d$ and $N \geq \left( \frac{\sinh 2k}{2k} \right)^{d-1} \left( \frac{\sqrt{2}\sinh k}{k} + 1 \right)^d$, then $V_{\mathbb{H}^d}(P^\delta) < V_{\mathbb{H}^d}(Q^\delta)$. 
On the intrinsic volumes of intersections of congruent balls

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Theorem 1. Let \( A \subset \mathbb{E}^d \), \( d > 1 \) be a compact set of volume \( V_d(A) > 0 \) and \( r > 0 \). If \( B \subset \mathbb{E}^d \) is a ball with \( V_d(A) = V_d(B) \), then

\[
V_k(A^r) \leq V_k(B^r)
\]

holds for all \( k \in [d] \).

Theorem 2. Let \( d > 1 \), \( \lambda > 0 \), \( r > 0 \), and \( k \in [d] \) be given and let \( Q := \{q_1, \ldots, q_N\} \subset \mathbb{E}^d \) be a uniform contraction of \( P := \{p_1, \ldots, p_N\} \subset \mathbb{E}^d \) with separating value \( \lambda \) in \( \mathbb{E}^d \).

(i) If \( 1 < d < 42 \) and \( N \geq (1 + \sqrt{2})^d \), then

\[
V_k(P^r) \leq V_k(Q^r).
\]

(ii) If \( d \geq 42 \) and \( N \geq \frac{\sqrt{2}\lambda}{1+\sqrt{2}}(1 + \sqrt{2})^d + 1 \), then (5) holds.
On uniform contractions of balls in Minkowski spaces

Károly Bezdek


Abstract

Let $N$ balls of the same radius be given in a $d$-dimensional real normed vector space, i.e., in a Minkowski $d$-space. Then apply a uniform contraction to the centers of the $N$ balls without changing the common radius. Here a uniform contraction is a contraction where all the pairwise distances in the first set of centers are larger than all the pairwise distances in the second set of centers. The main results of this paper state that a uniform contraction of the centers does not increase (resp., decrease) the volume of the union (resp., intersection) of $N$ balls in Minkowski $d$-space, provided that $N \geq 2^d$ (resp., $N \geq 3^d$ and the unit ball of the Minkowski $d$-space is a generating set). Some improvements are presented in Euclidean spaces.
Core notions and notations:

Let \( K \subseteq \mathbb{R}^d \) be an \( o \)-symmetric convex body, i.e., a compact convex set with nonempty interior symmetric about the origin \( o \) in \( \mathbb{R}^d \). Let \( \| \cdot \|_K \) denote the norm generated by \( K \), which is defined by \( \| x \|_K := \min\{\lambda \geq 0 \mid \lambda x \in K\} \) for \( x \in \mathbb{R}^d \). Furthermore, let us denote \( \mathbb{R}^d \) with the norm \( \| \cdot \|_K \) by \( \mathbb{M}^d_K \) and call it the Minkowski space of dimension \( d \) generated by \( K \). We write \( B_K[x, r] := x + rK \) for \( x \in \mathbb{R}^d \) and \( r > 0 \) and call any such set a (closed) ball of radius \( r \), where \( + \) refers to vector addition extended to subsets of \( \mathbb{R}^d \) in the usual way. The following definitions introduce the core notions and notations for our paper.

**Definition 1.** For \( X \subseteq \mathbb{R}^d \) and \( r > 0 \) let

\[
X_r^K := \bigcup\{B_K[x, r] \mid x \in X\} \quad \text{(resp.,) \quad X_r^r := \bigcap\{B_K[x, r] \mid x \in X\)}
\]

denote the \( r \)-ball neighbourhood of \( X \) (resp., \( r \)-ball body generated by \( X \)) in \( \mathbb{M}^d_K \). If \( X \subseteq \mathbb{R}^d \) is a finite set, then we call \( X_r^K \) (resp., \( X_r^r \)) the \( r \)-ball molecule (resp., \( r \)-ball polyhedron) generated by \( X \) in \( \mathbb{M}^d_K \).

**Definition 2.** We say that the (labeled) point set \( Q := \{q_1, \ldots, q_N\} \subseteq \mathbb{R}^d \) is a uniform contraction of the (labeled) point set \( P := \{p_1, \ldots, p_N\} \subseteq \mathbb{R}^d \) with separating value \( \lambda > 0 \) in \( \mathbb{M}^d_K \) if

\[
\|q_i - q_j\|_K \leq \lambda \leq \|p_i - p_j\|_K \quad \text{holds for all} \quad 1 \leq i < j \leq N.
\]
Monotonicity of the volume of r-ball molecules under uniform contractions of the centers in Minkowski spaces

In order to state the main results of this paper, let $V_d(\cdot)$ denote the Lebesgue measure in $\mathbb{R}^d$ (with $V_d(\emptyset) = 0$).

**Theorem 2.** Let $K$ be an o-symmetric convex body in $\mathbb{R}^d$. If $r > 0, \lambda > 0, d > 1$, $N \geq 2^d$, and $Q := \{q_1, \ldots, q_N\} \subset \mathbb{R}^d$ is a uniform contraction of $P := \{p_1, \ldots, p_N\} \subset \mathbb{R}^d$ with separating value $\lambda$ in $\mathbb{M}^d_K$, then

$$V_d(Q^K_r) \leq V_d(P^K_r).$$

**Remark 3.** The proof of Theorem 2 presented below yields the following statement as well. If $r \geq \frac{\lambda}{2} > 0$, $d > 1$, $N \geq 2^d$, and $Q$ is a uniform contraction of $P$ with separating value $\lambda$ in $\mathbb{M}^d_K$, then

$$V_d(\text{conv}(Q^K_r)) \leq V_d(\text{conv}(P^K_r)),$$

where $\text{conv}(\cdot)$ stands for the convex hull of the given set in $\mathbb{R}^d$. 
2 Proof of Theorem 2

As (1) holds trivially for $0 < r \leq \frac{\lambda}{2}$ therefore we may assume that $0 < \frac{\lambda}{2} < r$. Recall that for a bounded set $\emptyset \neq X \subset \mathbb{R}^d$ the diameter $\text{diam}_K(X)$ of $X$ in $M^d_K$ is defined by $\text{diam}_K(X) := \sup \{ \|x_1 - x_2\|_K \mid x_1, x_2 \in X \}$. Clearly,

$$\text{diam}_K(Q_r^K) = \text{diam}_K(Q) + 2r \leq \lambda + 2r. \quad (9)$$

Applying the isodiametric inequality:

Thus, the isodiametric inequality in Minkowski spaces (Theorem 11.2.1 in [9]) and (9) imply that

$$V_d(Q_r^K) \leq \left( r + \frac{\lambda}{2} \right)^d V_d(K). \quad (10)$$

Applying the isoperimetric inequality:

For the next estimate recall that the *volumetric radius relative to* $K$ of the compact set $\emptyset \neq A \subset \mathbb{R}^d$ is denoted by $r_K(A)$ and it is defined by $V_d\left( r_K(A)K \right) = (r_K(A))^d V_d(K) := V_d(A)$. Using this concept one can derive the following inequality from the Brunn–Minkowski inequality in a rather straightforward way (Theorem 9.1.1 in [9]):

$$r_K(A^K_{\epsilon}) \geq r_K(A) + \epsilon,$$

which holds for any $\epsilon > 0$. As $\{B_K[p_i, \lambda/2] \mid 1 \leq i \leq N\}$ is a packing in $\mathbb{R}^d$ therefore $r_K\left( P^K_{\lambda/2} \right) = N^{\frac{d}{2}} \lambda^\frac{1}{2}$.

Combining this observation with (11) yields

$$V_d(P^K_r) = V_d\left( \left( P^K_{\frac{\lambda}{2}} \right)_{r-\frac{\lambda}{2}} \right) \geq \left( \frac{N^\frac{d}{2}}{2} + (r - \frac{\lambda}{2}) \right)^d V_d(K) = \left( r + (N^\frac{1}{2} - 1) \frac{\lambda}{2} \right)^d V_d(K).$$

Finally, as $N \geq 2^d$ therefore $N^{\frac{1}{2}} - 1 \geq 1$ and Theorem 2 follows from (10) and (12) in a straightforward way.
Recall from [28] that the compact convex set $\emptyset \neq A' \subset \mathbb{R}^d$ is a summand of the compact convex set $\emptyset \neq A \subset \mathbb{R}^d$ if there exists a compact convex set $\emptyset \neq A'' \subset \mathbb{R}^d$ such that $A' + A'' = A$. Furthermore, following [23] we say that the convex body $B \subset \mathbb{R}^d$ is a generating set if any nonempty intersection of translates of $B$ is a summand of $B$. In particular, we say that $M^d_K$ possesses a generating unit ball if $B_K[0, 1] = K$ is a generating set in $\mathbb{R}^d$. For a recent overview on generating sets see the relevant subsections in [22] and [23]. Here we recall the following statements only. Two-dimensional convex bodies are generating sets. Euclidean balls are generating sets as well and the system of generating sets is stable under non-degenerate linear maps and under direct sums. Furthermore, a centrally symmetric convex polytope is a generating set if and only if it is a direct sum of convex polygons and in odd dimension, a line segment.

**Theorem 4.** Let $r > 0, \lambda > 0, d > 1, N \geq 3^d$, and let the $o$-symmetric convex body $K$ be a generating set in $\mathbb{R}^d$. If $Q := \{q_1, \ldots, q_N\} \subset \mathbb{R}^d$ is a uniform contraction of $P := \{p_1, \ldots, p_N\} \subset \mathbb{R}^d$ with separating value $\lambda$ in $M^d_K$, then

$$V_d(P_K^r) \leq V_d(Q_K^r).$$
On maximizing the volume of r-ball bodies generated by sets of given volume

Remark 5. We say that the balls of $\mathbb{M}^d_K$ are volumetric maximizers for r-ball bodies in $\mathbb{M}^d_K$ if for any compact set $\emptyset \neq A \subset \mathbb{R}^d$ with $V_d(A) > 0$ the inequality

$$V_d(A^T_K) \leq V_d(B_K[0, r - r_K(A)])$$

holds for all $r > r_K(A)$, where $V_d(A) = V_d(B_K[0, r_K(A)])$. On the one hand, if the balls of $\mathbb{M}^d_K$ are generating sets in $\mathbb{R}^d$, then they are volumetric maximizers for r-ball bodies in $\mathbb{M}^d_K$. On the other hand, if the balls of $\mathbb{M}^d_K$ are volumetric maximizers for r-ball bodies in $\mathbb{M}^d_K$, then (3) holds whenever $Q := \{q_1, \ldots, q_N\} \subset \mathbb{R}^d$ is a uniform contraction of $P := \{p_1, \ldots, p_N\} \subset \mathbb{R}^d$ with separating value $\lambda$ in $\mathbb{M}^d_K$ and $r > 0, \lambda > 0, d > 1, N \geq 3^d$. Thus, it would be interesting to find a proper characterization of those Minkowski spaces $\mathbb{M}^d_K$ whose balls are volumetric maximizers for r-ball bodies, that is, for which (4) holds.
4 Proof of Theorem 4

The following proof extends the core ideas of the proof of Theorem 1.4 from [7] to Minkowski spaces. For a bounded set $\emptyset \neq X \subset \mathbb{R}^d$ let $\operatorname{cr}_K(X) := \inf\{R > 0 \mid X \subseteq B_K[x, R] \text{ with } x \in \mathbb{R}^d\}$. We call $\operatorname{cr}_K(X)$ the circumradius of $X$ in $\mathbb{M}^d_K$. Now, recall that $P = \{p_1, \ldots, p_N\} \subset \mathbb{R}^d$ with $N \geq 3^d$ such that $\lambda \leq \|p_i - p_j\|_K$ holds for all $1 \leq i < j \leq N$. We claim that

$$\lambda \leq \operatorname{cr}_K(P). \tag{15}$$

If $r \leq \operatorname{cr}_K(P)$, then clearly $V_d(P^r_K) = V_d(\emptyset) = 0 \leq V_d(Q^r_K)$, finishing the proof of Theorem 4 in this case. Hence, for the rest of the proof of Theorem 4 we may assume via (15) that

$$0 < \lambda \leq \operatorname{cr}_K(P) < r. \tag{18}$$

Applying Bohnenblust’s theorem:

Next, recall that $Q = \{q_1, \ldots, q_N\} \subset \mathbb{R}^d$ with $N \geq 3^d$ such that $\|q_i - q_j\|_K \leq \lambda$ holds for all $1 \leq i < j \leq N$. Thus, Bohnenblust’s theorem (Theorem 11.1.3 in [9]) yields $\operatorname{cr}_K(Q) \leq \frac{d}{d+1} \operatorname{diam}_K(Q) \leq \frac{d}{d+1} \lambda$, from which it is easy to derive that

$$V_d(Q^r_K) \geq \left(r - \frac{d}{d+1} \lambda \right)^d V_d(K). \tag{19}$$
Relating r-ball body to r-ball convex hull:

For a bounded set \( \emptyset \neq X \subset \mathbb{R}^d \) and \( r > 0 \) with \( \operatorname{cr}_K(X) \leq r \) let \( \operatorname{conv}_{r,K}(X) := \bigcap \{ B_K[x, r] \mid X \subset B_K[x, r] \text{ with } x \in \mathbb{R}^d \} \). We call \( \operatorname{conv}_{r,K}(X) \) the \( r \)-ball convex hull of \( X \) in \( \mathbb{M}^d_K \). If \( \emptyset \neq X \subset \mathbb{R}^d \) is a bounded set and \( r > 0 \) with \( \operatorname{cr}_K(X) > r \), then let \( \operatorname{conv}_{r,K}(X) := \mathbb{R}^d \). Moreover, for an unbounded set \( X \subset \mathbb{R}^d \) and \( r > 0 \) let \( \operatorname{conv}_{r,K}(X) := \mathbb{R}^d \). Furthermore, for simplicity let \( \operatorname{conv}_{r,K}(\emptyset) := \emptyset \). Finally, we say that \( X \subset \mathbb{R}^d \) is \( r \)-ball convex for \( r > 0 \) in \( \mathbb{M}^d_K \) if \( X = \operatorname{conv}_{r,K}(X) \). Clearly, \( X^r_K \) is \( r \)-ball convex in \( \mathbb{M}^d_K \) for any \( X \subset \mathbb{R}^d \).

**Lemma 8.** Let \( d > 1 \) and \( r > 0 \) be given and let \( \mathbb{M}^d_K \) possess a generating unit ball. If \( X^r_K \neq \emptyset \), then

\[
X^r_K - \operatorname{conv}_{r,K}(X) = B_K[0, r].
\]  

**Proof.** Clearly, as \( B_K[0, 1] = K \) is a generating set in \( \mathbb{R}^d \) therefore the closed ball having radius \( r > 0 \) in \( \mathbb{M}^d_K \), i.e., \( B_K[0, r] = rK \) is also a generating set in \( \mathbb{R}^d \). In particular, \( X^r_K \neq \emptyset \) is a summand of \( B_K[0, r] \). Now, recall Lemma 3.1.8 of [28] stating that the compact convex set \( \emptyset \neq A' \subset \mathbb{R}^d \) is a summand of the compact convex set \( \emptyset \neq A \subset \mathbb{R}^d \) if and only if \( (A \sim A') + A' = A \), where \( A \sim A' := \bigcap_{a' \in A'} (A - a') \). This implies that \( (B_K[0, r] \sim X^r_K) + X^r_K = B_K[0, r] \). Finally, we are left to observe that \( B_K[0, r] \sim X^r_K = \bigcap_{x \in X^r_K} (B_K[0, r] - x) = -\bigcap_{x \in X^r_K} B_K[x, r] = -\operatorname{conv}_{r,K}(X) \), finishing the proof of Lemma 8. \( \square \)
Applying the Brunn-Minkowski inequality:

Clearly, the Brunn–Minkowski inequality ([9], [28]) combined with Lemma 8 yields

**Corollary 10.** Let \( d > 1 \) and \( r > 0 \) be given and let \( M_d^d \) possess a generating unit ball. If \( X_r^d_k \neq \emptyset \), then

\[
V_d(X_r^d_k)^{\frac{1}{d}} + V_d(\text{conv}_{r+\frac{1}{2}}\,X_r^d_k(X))^\frac{1}{d} \leq r V_d(K)^{\frac{1}{d}}. \tag{21}
\]

Next, observe that based on (18) we have \( \emptyset \neq P_r^d = \left( P^d_k \right)^{r+\frac{1}{2}} \) and so, Corollary 10 yields

\[
V_d(P_r^d) = V_d \left( \left( P^d_k \right)^{r+\frac{1}{2}} \right) \leq \left( \left( r + \frac{\lambda}{2} \right) V_d(K)^{\frac{1}{d}} - V_d \left( \text{conv}_{r+\frac{1}{2}}\,P^d_k \right)^{\frac{1}{d}} \right)^d \leq \left( r - (N^d - 1) \frac{\lambda}{2} \right)^d V_d(K), \tag{22}
\]

where in the last inequality we have used the fact that \( \{ B_k \left[ p_i, \frac{\lambda}{2} \right] \mid 1 \leq i \leq N \} \) is a packing in \( \mathbb{R}^d \) and therefore \( V_d \left( \text{conv}_{r+\frac{1}{2}}\,P^d_k \right) \geq N \left( \frac{\lambda}{2} \right)^d V_d(K) \). Finally, observe that \( N \geq 3^d \) implies \( \left( r - (N^d - 1) \frac{\lambda}{2} \right)^d V_d(K) \leq (r - \lambda)^d V_d(K) < \left( r - \frac{d}{d+1} \lambda \right)^d V_d(K) \). This inequality combined with (19) and (22) completes the proof of Theorem 4.
Improvements in Euclidean spaces

Let \( \emptyset \neq A \subset \mathbb{E}^d \) be a compact convex set, and \( 0 \leq k \leq d \). We denote the \( k \)-th quermassintegral of \( A \) by \( W_k(A) \). It is well known that \( W_0(A) = V_d(A) \). Moreover, \( dW_1(A) \) is the surface area of \( A \), \( \frac{2}{\omega_d} W_{d-1}(A) \) is equal to the mean width of \( A \), and \( W_d(A) = \omega_d \), where \( \omega_d \) stands for the volume of a \( d \)-dimensional unit ball, that is, \( \omega_d = \frac{\pi^{\frac{d}{2}}}{\Gamma(1+\frac{d}{2})} \) ([28], p. 290-291). In this paper, for simplicity \( W_k(\emptyset) = 0 \) for all \( 0 \leq k \leq d \). Here we recall Kubota’s integral recursion formula ([28], p. 295), according to which

\[
W_k(A) = \frac{1}{d\omega_{d-1}} \int_{S_{d-1}} W_{k-1}(P_{u \perp}(A)) \, d\lambda(u)
\]  

(5)

holds for any compact convex set \( \emptyset \neq A \subset \mathbb{E}^d \) and for any \( 0 < k < d \), where \( S^{d-1} := \text{bd}(B^d[0,1]) = \{ x \in \mathbb{E}^d \mid |x| = 1 \} \), \( d\lambda(\cdot) \) is the spherical Lebesgue measure on \( S^{d-1} \), and \( P_{u \perp}(\cdot) \) is the orthogonal projection onto the orthogonal complement of the 1-dimensional linear subspace spanned by \( u \in S^{d-1} \). Finally, we recall that Ohmann [24], [25], [26] using Kubota’s formula (5) has inductively defined the quermassintegrals \( W_k(A) \), \( 0 < k < d \) for any compact set \( \emptyset \neq A \subset \mathbb{E}^d \) with \( W_0(A) := V_d(A) \) and \( W_d(A) := \omega_d \) and proved analogues of some classical inequalities on quermassintegrals. In what follows we use Ohmann’s extension of the classical quermassintegrals for non-convex compact sets.

Improving Theorems 2 and 4 in Euclidean spaces:

**Theorem 2.** Let $K$ be an o-symmetric convex body in $\mathbb{R}^d$. If $r > 0, \lambda > 0, d > 1$, and $Q := \{q_1, \ldots, q_N\} \subset \mathbb{R}^d$ is a uniform contraction of $P := \{p_1, \ldots, p_N\} \subset \mathbb{R}^d$ with separating value $\lambda$ in $M^d_K$, then

$$V_d(Q^k_r) \leq V_d(P^k_r).$$

(1)

**Theorem 4.** Let $r > 0, \lambda > 0, d > 1$, and let the o-symmetric convex body $K$ be a generating set in $\mathbb{R}^d$. If $Q := \{q_1, \ldots, q_N\} \subset \mathbb{R}^d$ is a uniform contraction of $P := \{p_1, \ldots, p_N\} \subset \mathbb{R}^d$ with separating value $\lambda$ in $M^d_K$, then

$$V_d(P^k_r) \leq V_d(Q^k_r).$$

(3)

**Theorem 6.**

(i) If $r \geq \frac{\lambda}{2} > 0, 0 \leq k < d$, $N \geq 2^d$, and $Q := \{q_1, \ldots, q_N\} \subset \mathbb{R}^d$ is a uniform contraction of $P := \{p_1, \ldots, p_N\} \subset \mathbb{R}^d$ with separating value $\lambda$ in $\mathbb{E}^d$, then

$$W_k(\text{conv}(Q^r_r)) \leq W_k(\text{conv}(P^r_r)).$$

(6)

and

$$W_k(Q^r_r) \leq W_k(P^r_r).$$

(7)

(ii) If $r > 0, \lambda > 0, d \geq d_0$ (with a (large) universal constant $d_0$), $N \geq 2.359^d$, and $Q := \{q_1, \ldots, q_N\} \subset \mathbb{R}^d$ is a uniform contraction of $P := \{p_1, \ldots, p_N\} \subset \mathbb{R}^d$ with separating value $\lambda$ in $\mathbb{E}^d$, then

$$V_d(P^r_r) \leq V_d(Q^r_r).$$

(8)
Proof of Part (ii) of Theorem 6

Recall that \( P := \{p_1, \ldots, p_N\} \subset \mathbb{E}^d \) such that \( 0 < \lambda \leq |p_i - p_j| \) holds for all \( 1 \leq i < j \leq N \), where \( N \geq 2.359^d \) with \( d \) being sufficiently large. We denote the circumradius of a set \( X \subset \mathbb{E}^d \), \( d > 1 \) by \( \text{cr}(X) \), which is defined by \( \text{cr}(X) := \inf\{r \mid X \subseteq B^d(x, r) \text{ for some } x \in \mathbb{E}^d\} \).

Lemma 11. \( \sqrt{\frac{2d}{d+1}} \left( \frac{\lambda}{2} \right) < 0.7865 \cdot \lambda < \text{cr}(P) \), where \( d \geq d_0 \) with a large universal constant \( d_0 > 0 \) and \( \text{card}(P) = N \geq 2.359^d \).

Proof. First, we note that \( B^d[p_1, \frac{\lambda}{2}], \ldots, B^d[p_N, \frac{\lambda}{2}] \) are pairwise non-overlapping in \( \mathbb{E}^d \). Thus, the Lemma of [2] and \( N \geq 2.359^d \) imply that

\[
\frac{2.359^d \left( \frac{\lambda}{2} \right)^d}{(\text{cr}(P) + \lambda)^d} \leq \frac{N \left( \frac{\lambda}{2} \right)^d}{(\text{cr}(P) + \lambda)^d} \leq \frac{V_d \left( \bigcup_{i=1}^N B^d[p_i, \frac{\lambda}{2}] \right)}{V_d \left( \bigcup_{i=1}^N B^d[p_i, \lambda] \right)} \leq \delta_d,
\]

(31)

where \( \delta_d \) stands for the largest density of packings of congruent balls in \( \mathbb{E}^d \). Second, recall that Kabatiansky and Levenshtein ([11]) have shown that

\[
\delta_d < 2^{-0.599d}
\]

(32)

holds for sufficiently large \( d \) say, for \( d \geq d_0 \), where \( d_0 > 0 \) is a large universal constant. Hence, the statement follows from (31) and (32) in a straightforward way. \( \square \)


If \( r \leq \text{cr}(P) \), then \( V_d(P^r) = V_d(\emptyset) = 0 \) and so, \( V_d(P^r) \leq V_d(Q^r) \), i.e., (8) follows. Thus, for the rest of the proof we assume that \( \text{cr}(P) < r \), which together with Lemma 11 implies

\[
\sqrt{\frac{2d}{d+1} \left( \frac{\lambda}{2} \right)} < 0.7865 \cdot \lambda < \text{cr}(P) < r
\]

(33)

with \( d \geq d_0 \) and \( \text{card}(P) = N \geq 2.359^d \). Next, as Euclidean balls are generating sets therefore (22) implies the following statement. (See also Lemma 2.6 of [7] and (18) in [8].)

\[
V_d(P^r_K) = V_d \left( \left( p^r_K \right)^{\frac{r}{2}} \right) \leq \left[ \left( r + \frac{\lambda}{2} \right) V_d(K)^{\frac{r}{d}} - V_d \left( \text{conv}_{r+\frac{\lambda}{2},K} \left( p^r_K \right) \right)^{\frac{r}{d}} \right] \leq \left( r - (N^{\frac{1}{d}} - 1) \frac{\lambda}{2} \right) V_d(K),
\]

(22)

Lemma 12. If \( d > 1, \lambda > 0, r > 0, \) and \( \text{card}(P) = N > 1 \), then \( V_d(P^r) \leq V_d \left( \mathbb{B}^d \left[ o, r - \left( N^{\frac{1}{d}} - 1 \right) \left( \frac{\lambda}{2} \right) \right] \right) \).

Here we follow the convention that if \( r - \left( N^{\frac{1}{d}} - 1 \right) \left( \frac{\lambda}{2} \right) < 0 \), then \( \mathbb{B}^d \left[ o, r - \left( N^{\frac{1}{d}} - 1 \right) \left( \frac{\lambda}{2} \right) \right] = \emptyset \) with \( V_d(\emptyset) = 0 \).
Lemma 13. $V_d(Q^r) \geq V_d\left(B^d\left[o, \sqrt{r^2 - \frac{d-1}{d+1} \left(\frac{\lambda}{2}\right)^2 - \left(\frac{\lambda}{2}\right)}\right]\right)$, where $d \geq d_0$ and $N \geq 2.359^d$.

Proof. First, recall Theorem 2 of [29].


Theorem 14. Let $K$ be a set of diameter $\sigma$ and circumradius $\rho$ in $\mathbb{E}^d$. If $\mu > \rho > 0$, then

$$V_d(K^\mu) \geq F\left(\mu, \rho, \frac{\sigma}{2}\right)^d \omega_d,$$

(34)

where $F(\mu, \rho, x) := \sqrt{\mu^2 - \rho^2 + x^2} - x$, which is a positive, decreasing, and convex function of $x > 0$.

Second, Jung’s theorem ([15]) implies that $cr(Q) \leq \sqrt{\frac{2d}{d+1}} \left(\frac{1}{2}\right)$ and (33) guarantees that $\sqrt{\frac{2d}{d+1}} \left(\frac{1}{2}\right) < r$.

Hence, from this and (34), using the monotonicity of $F(\mu, \rho, x)$ in $x$ (resp., $\rho$), one obtains

$$V_d\left(B^d\left[o, \sqrt{r^2 - \frac{d-1}{d+1} \left(\frac{\lambda}{2}\right)^2 - \left(\frac{\lambda}{2}\right)}\right]\right) = V_d\left(B^d\left[o, F\left(r, \sqrt{\frac{2d}{d+1}} \left(\frac{\lambda}{2}\right), \left(\frac{\lambda}{2}\right)\right)\right]\right) \leq V_d(Q^r),$$

(35)

which completes the proof of Lemma 13. \qed
Clearly, Lemma 12 and Lemma 13 imply that in order to show the inequality $V_d(P^r) \leq V_d(Q^r)$, it is sufficient to prove

$$r - \left( N^{\frac{1}{d}} - 1 \right) \left( \frac{\lambda}{2} \right) \leq \sqrt{r^2 - \frac{d - 1}{d + 1} \left( \frac{\lambda}{2} \right)^2} - \left( \frac{\lambda}{2} \right).$$

(36)

(36) is equivalent to

$$\left( \frac{2r}{\lambda} \right) - \sqrt{\left( \frac{2r}{\lambda} \right)^2 - \frac{d - 1}{d + 1}} + 2 \leq N^{\frac{1}{d}}$$

(37)

and obviously, (37) follows (via $d \geq d_0$ and $N \geq 2.359^d$) from

$$\left( \frac{2r}{\lambda} \right) - \sqrt{\left( \frac{2r}{\lambda} \right)^2} - 1 + 2 \leq 2.359.$$  

(38)

Finally, as $f(x) := x - \sqrt{x^2 - 1}$ is a positive and decreasing function for $x > 1$ and as (33) guarantees that $1.573 < \frac{2r}{\lambda}$ therefore (38) follows from $1.573 - \sqrt{1.573^2 - 1} + 2 = 2.3587... < 2.359$. This completes the proof of Theorem 6.