Local Search

Nabil H. Mustafa
Expansion in Graphs

Let $G = (B, R, E)$ be a bipartite graph such that

$$\forall B' \subseteq B, \quad |B'| \leq k \quad \implies \quad |\text{Neighbors}(B')| \geq |B'|$$
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$G$ is $k$-expanding
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‘locally’ the size of $R$ is at least that of $B$
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\( G \) is \( k \)-expanding

‘locally’ the size of \( R \) is at least that of \( B \)

**Question:** does this imply that \( |R| \geq |B|? \quad \sqrt{|B|}? \)
Expansion in Graphs

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**Question:** does this imply that $|R| \geq |B|$? $\sqrt{|B|}$?
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$G$ is $k$-expanding

‘locally’ the size of $R$ is at least that of $B$

**Question:** does this imply that $|R| \geq |B|? \quad \sqrt{|B|}?$

$$|R| \geq k$$
Expansion in Planar Graphs

Let $G = (B, R, E)$ be a planar graph such that

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**Question:** $|R| \geq \cdots$
Let $G = (B, R, E)$ be a planar graph such that

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**Question:** $|R| \geq \cdots$

$$k = 1$$
Expansion in Planar Graphs

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**Question:** $|R| \geq \cdots$

$k = 1$

$k = 2$
Expansion in Planar Graphs

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**Question:** $|R| \geq \cdots$

$k = 1$

$k = 2$

$k = 3$
Let $G = (B, R, E)$ be a planar graph such that
\[
\forall B' \subseteq B, \quad |B'| \leq k \quad \implies \quad |\text{Neighbors}(B')| \geq |B'|
\]

**Question:** $|R| \geq \cdots$

$k = 1$

$k = 2$

$k = 3$
Expansion in Planar Graphs
Expansion in Planar Graphs

3-expanding
3-expanding

$$|R| \leq \frac{|B|}{8}$$
3-expanding

$$|R| \leq \frac{|B|}{8}$$

**Theorem:** Any 3-expanding planar bipartite graph has $|R| \geq \frac{|B|}{8}$.

Expansion in Planar Graphs
4-expanding
4-expanding

\[ |R| \leq \frac{|B|}{4} \]
Expansion in Planar Graphs

4-expanding

\[ |R| \leq \frac{|B|}{4} \]

Theorem: Any 4-expanding planar bipartite graph has \( |R| \geq \frac{|B|}{4} \).

[Antunes, Mathieu, M. 2017.]
Let $G = (B, R, E)$ be a planar graph such that

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Expansion in Planar Graphs

Let $G = (B, R, E)$ be a planar graph such that

$$\forall B' \subseteq B, \quad |B'| \leq k \quad \implies \quad |\text{Neighbors}(B')| \geq |B'|$$

Theorem: \hfill $|R| \geq \left(1 - O\left(\frac{1}{\sqrt{k}}\right)\right) |B|$
Let $G = (B, R, E)$ be a planar graph such that

$$\forall B' \subseteq B, \quad |B'| \leq k \quad \Longrightarrow \quad |\text{Neighbors}(B')| \geq |B'|$$

**Theorem:**

$$|R| \geq \left(1 - O\left(\frac{1}{\sqrt{k}}\right)\right)|B|$$

for any integer $k$
Expansion in Planar Graphs

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Lipton and Tarjan, 1979
Expansion in Planar Graphs

Let $G = (B, R, E)$ be a planar graph such that

$$\forall B' \subseteq B, \quad |B'| \leq k \quad \iff \quad |\text{Neighbors}(B')| \geq |B'|$$

**Theorem:**

$$|R| \geq \left(1 - O\left(\frac{1}{\sqrt{k}}\right)\right) |B|$$

for any integer $k$

**Claim:**

$$|R_1| \geq |B_1| - \sqrt{k}.$$
Expansion in Planar Graphs

Let $G = (B, R, E)$ be a planar graph such that

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**Claim:**

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Iterating $\frac{|R| + |B|}{k}$ times,

$$\leq \sqrt{k}$$
Expansion in Planar Graphs

Let $G = (B, R, E)$ be a planar graph such that

$$\forall B' \subseteq B, \quad |B'| \leq k \quad \Rightarrow \quad |\text{Neighbors}(B')| \geq |B'|$$

**Theorem:** \( |R| \geq \left(1 - O\left(\frac{1}{\sqrt{k}}\right)\right) |B| \)

**Claim:** \( |R_1| \geq |B_1| - \sqrt{k} \).

Iterating \( \frac{|R| + |B|}{k} \) times,

$$\sum_{i=1}^{\frac{|R| + |B|}{k}} |R_i| \geq \left( \sum_{i=1}^{\frac{|R| + |B|}{k}} |B_i| \right) - \frac{|R| + |B|}{k} \sqrt{k}$$

\( R_1 \quad B_1 \quad k \)

\( \leq \sqrt{k} \)
Expansion in Planar Graphs

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Claim: $|R_1| \geq |B_1| - \sqrt{k}$.

Iterating $\frac{|R| + |B|}{k}$ times,

$$\sum_{i=1}^{\frac{|R| + |B|}{k}} |R_i| \geq \left(\sum_{i=1}^{\frac{|R| + |B|}{k}} |B_i|\right) - \frac{|R| + |B|}{k} \sqrt{k}$$

$$\implies |R| \geq |B| \cdot \left(1 - \frac{20}{\sqrt{k}}\right).$$
INDEPENDENT SETS IN PLANAR GRAPHS
Independent Sets in Planar Graphs
INDEPENDENT SETS IN PLANAR GRAPHS
Start with any independent set $I \subseteq V$

While (possible) {

$\text{Replace } k - 1 \text{ vertices in } I \text{ with } k \text{ vertices of } V \setminus I$

$\text{while maintaining } I \text{ to be an independent set}$

}
Independent Sets in Planar Graphs

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$I$: output of the $k$-local search algorithm.
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Independent Sets in Planar Graphs

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}\}

\( I \): output of the \( k \)-local search algorithm.

\( O \): a maximum independent set

\[ \text{Goal: } |I| \geq (1 - \delta_k) \cdot |O| \]
Observation:
Observation:

$I$: a $k$-locally optimal solution
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Take any $O' \subseteq O$ of size at most $k$
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$I$: a $k$-locally optimal solution

Take any $O' \subseteq O$ of size at most $k$

$I - N(O') + O'$ is an independent set
**Independent Sets in Planar Graphs**

**Observation:**

$I$: a $k$-locally optimal solution

Take any $O' \subseteq O$ of size at most $k$

$I - N(O') + O'$ is an independent set

$\implies |N(O')| \geq |O'|$

local optimality $\implies$ expansion in planar graphs
Independent Sets in Planar Graphs

\( G' \): planar bipartite graph on \( I \) and \( O \).

\[
\forall O' \subseteq O, \quad |O'| \leq k \quad \implies \quad |\text{Neighbors}(O')| \geq |O'|
\]
**Independent Sets in Planar Graphs**

$G'$: planar bipartite graph on $I$ and $O$.

$$\forall O' \subseteq O, \quad |O'| \leq k \quad \implies \quad |\text{Neighbors}(O')| \geq |O'|$$

**Theorem:** $$|I| \geq \left(1 - O\left(\frac{1}{\sqrt{k}}\right)\right) |O|$$
\( G' \): planar bipartite graph on \( I \) and \( O \).

\[ \forall O' \subseteq O, \quad |O'| \leq k \quad \implies \quad |\text{Neighbors}(O')| \geq |O'| \]

**Theorem:** \[ |I| \geq \left( 1 - O\left( \frac{1}{\sqrt{k}} \right) \right) |O| \]

If \( I \) cannot be improved by adding/removing some \( O \left( \frac{1}{\epsilon^2} \right) \) vertices
Independent Sets in Planar Graphs

$G'$: planar bipartite graph on $I$ and $O$.

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**Theorem:** \[ |I| \geq \left(1 - O\left(\frac{1}{\sqrt{k}}\right)\right)|O| \]

If $I$ cannot be improved by adding/removing some $O\left(\frac{1}{\epsilon^2}\right)$ vertices

\[ \implies |I| \geq (1 - \epsilon)|O| \]
\textbf{Independent Sets in Planar Graphs}

$G'$: planar bipartite graph on $I$ and $O$.

\[ \forall O' \subseteq O, \quad |O'| \leq k \quad \implies \quad |\text{Neighbors}(O')| \geq |O'| \]

\begin{center}
\textbf{Theorem:} \quad |I| \geq \left(1 - O\left(\frac{1}{\sqrt{k}}\right)\right) |O| \end{center}

If $I$ cannot be improved by adding/removing some $O\left(\frac{1}{\epsilon^2}\right)$ vertices

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\textbf{Corollary:} $\Theta\left(\frac{1}{\epsilon^2}\right)$-local search gives PTAS for M.I.S. in planar graphs
**Independent Sets in Planar Graphs**

$G'$: planar bipartite graph on $I$ and $O$.

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**Theorem:**

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|I| \geq \left(1 - O\left(\frac{1}{\sqrt{k}}\right)\right) |O|
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**Corollary:** $\Theta\left(\frac{1}{\epsilon^2}\right)$-local search gives PTAS for M.I.S. in planar graphs

**Can this be improved?**
A Lower-Bound

Theorem: For any integer $n$ and $k$, there exists a planar bipartite graph $G = (B, R, E)$ such that $G$ is $k$-expanding and $|R| = \Theta \left( \left( 1 - O\left( \frac{1}{\sqrt{k}} \right) \right) |B| \right)$
A Lower-Bound

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A Lower-Bound

**Theorem:** For any integer $n$ and $k$, there exists a planar bipartite graph $G = (B, R, E)$ such that $G$ is $k$-expanding and $|R| = \Theta \left( \left(1 - O\left(\frac{1}{\sqrt{k}}\right)\right)|B|\right)$.
A Lower-Bound

$$|R| = n \quad |B| = n + \frac{n}{l^2}$$
A Lower-Bound

$$|R| = n \quad \quad |B| = n + \frac{n}{l^2}$$

For what value of $k$ is $G$ $k$-expanding?
A Lower-Bound

Fix any $B' \subseteq B$. 
A Lower-Bound

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A Lower-Bound

Fix any $B' \subseteq B$.

charge to a vertex of $B'$ to its left-bottom corner red vertex
A Lower-Bound

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**Boundary vertex:** a vertex of $N(B')$ whose cell is empty
A Lower-Bound

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**Claim:** at least $\sqrt{|B'|}$ boundary vertices
A Lower-Bound

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**Claim**: at least $\sqrt{|B'|}$ boundary vertices

**Claim**: at most $\frac{|B'|}{l^2}$ double vertices
Fix any $B' \subseteq B$.

**Boundary vertex:** a vertex of $N(B')$ whose cell is empty

**Claim:** at least $\sqrt{|B'|}$ boundary vertices

**Claim:** at most $\frac{|B'|}{\ell^2}$ double vertices

$$\frac{|B'|}{\ell^2} \leq \sqrt{|B'|}$$
Fix any $B' \subseteq B$.

**Boundary vertex:** a vertex of $N(B')$ whose cell is empty

**Claim:** at least $\sqrt{|B'|}$ boundary vertices

**Claim:** at most $\frac{|B'|}{l^2}$ double vertices

$$\frac{|B'|}{l^2} \leq \sqrt{|B'|}$$

$$\implies |B'| \leq l^4$$
A Lower-Bound

\[ |R| = n \quad |B| = n + \frac{n}{l^2} \]

\[ G \text{ is } \Theta(l^4)\text{-expanding} \]
A Lower-Bound

\[ |R| = n \quad |B| = n + \frac{n}{l^2} \]

\( G \) is \( \Theta (l^4) \)-expanding

Setting \( l = \Theta \left( k^{\frac{1}{4}} \right) \) gives the lower-bound.
\[ |R| = n \quad |B| = n + \frac{n}{l^2} \]

\[ G \text{ is} \quad \Theta \left( l^4 \right) \text{-expanding} \]

Setting \( l = \Theta \left( k^{\frac{1}{4}} \right) \) gives the lower-bound.

**Theorem:** For any integer \( n \) and \( k \), there exists a planar bipartite graph \( G = (B, R, E) \) such that \( G \) is \( k \)-expanding and \( |R| = \Theta \left( \left( 1 - O\left( \frac{1}{\sqrt{k}} \right) \right) |B| \right) \)

[Jartoux, M., 2018.]
Independent Sets in Planar Graphs
**Theorem:** $\Theta\left(\frac{1}{\varepsilon^2}\right)$-local search gives PTAS for M.I.S. in planar graphs. This cannot be improved.
Independent Sets in Planar Graphs

Lipton and Tarjan, 1980
Independent Sets in Planar Graphs

Lipton and Tarjan, 1980

\[ \tilde{O}(\sqrt{n}) \]

\[ \left[ \frac{n}{3}, \frac{2n}{3} \right] \]

\[ \left[ \frac{n}{3}, \frac{2n}{3} \right] \]
Divide-and-conquer using separators
Independent Sets in Planar Graphs

Lipton and Tarjan, 1980

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Divide-and-conquer using separators

1. compute a separator \( C \) for \( G \)
Divide-and-conquer using separators

1. compute a separator $C$ for $G$

2. throw away the vertices of $C$
Divide-and-conquer using separators

1. compute a separator $C$ for $G$
2. throw away the vertices of $C$
3. recursively solve the problem for interior and exterior subgraphs
Independent Sets in Planar Graphs

Lipton and Tarjan, 1980

\[ O(\sqrt{n}) \]

\[ \left[ \frac{n}{3}, \frac{2n}{3} \right] \]

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Independent Sets in Planar Graphs

Lipton and Tarjan, 1980

Key point:
Independent Sets in Planar Graphs

Lipton and Tarjan, 1980

Key point: planar graphs are sparse: $3n - 6$ edges
**Independent Sets in Planar Graphs**

Lipton and Tarjan, 1980

Key point:

*planar graphs are sparse:* $3n - 6$ edges

$\Rightarrow$ independent set: $\frac{n}{5}$
**Independent Sets in Planar Graphs**

Lipton and Tarjan, 1980

**Key point:**

- Planar graphs are sparse: \(3n - 6\) edges

\[\implies\text{independent set: }\frac{n}{5}\]

\[\implies\sqrt{n}\text{ is a ‘relatively small’ loss}\]
Key point:

planar graphs are sparse: $3n - 6$ edges

$\implies$ independent set: $\frac{n}{5}$

$\implies \sqrt{n}$ is a ‘relatively small’ loss

**Theorem.** there exists a PTAS to compute M.I.S. in planar graphs.
Independent Sets in Intersection Graphs
Independent Sets in Intersection Graphs

intersection graphs: $\Theta(n^2)$ edges
intersection graphs: \( \Theta(n^2) \) edges
**Independent Sets in Intersection Graphs**

**intersection graphs:** \( \Theta(n^2) \) edges

throwing away one wrong vertex can make PTAS impossible!
Independent Set Algorithm

Start with any independent set \( I \subseteq D \)

While (possible) {

Replace \( k - 1 \) disks in \( I \) with \( k \) disks of \( D \)
while maintaining \( I \) to be an independent set

}
Independent Set for Disks

Independent Set Algorithm

Start with any independent set \( I \subseteq \mathcal{D} \)

While (possible) {

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}\n
**immediate**: intersection graph of \( I \) and \( O \)
is \( k \)-expanding.
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}

**Immediate:** intersection graph of \( I \) and \( O \) is \( k \)-expanding.

**Claim:** intersection graph of \( I \) and \( O \) is planar.
Independent Set Algorithm

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intersection graph of \( \mathcal{D} \) is not planar!
Independent Set for Disks

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}

**immediate:** intersection graph of \( I \) and \( O \)
is \( k \)-expanding.

**Claim:** intersection graph of \( I \) and \( O \) is planar.

**intersection graph of \( D \) is not planar!**

**Corollary:** \( \Theta \left( \frac{1}{e^2} \right) \)-local search gives PTAS for M.I.S. in disk intersection graphs
**Local Search**

**Theorem.** $k$-local search gives a $\left(1 - O\left(\frac{1}{\sqrt{k}}\right)\right)$-approximation for ...
Local Search

**Theorem.** $k$-local search gives a $\left(1 - O\left(\frac{1}{\sqrt{k}}\right)\right)$-approximation for ...

set-cover, hitting set

... Delaunay graphs are planar

[M. and Ray, 2010.]
**Local Search**

**Theorem.** $k$-local search gives a $\left(1 - O\left(\frac{1}{\sqrt{k}}\right)\right)$-approximation for …

set-cover, hitting set

… Delaunay graphs are planar [M. and Ray, 2010.]

independent set

… bipartite intersection graphs are planar [Chan and Har-Peled, 2012]
Local Search

Theorem. \( k \)-local search gives a \( \left( 1 - O\left(\frac{1}{\sqrt{k}}\right) \right) \)-approximation for ...

set-cover, hitting set
  ... Delaunay graphs are planar \[ \text{[M. and Ray, 2010.]} \]

independent set
  ... bipartite intersection graphs are planar \[ \text{[Chan and Har-Peled, 2012]} \]

terrain guarding
  ... visibility graph is planar \[ \text{[Krohn et al. 2014.]} \]
Local Search

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minor-free graph families
  ... expansion theorem holds [Cabello and Gajser, 2015.]
Local Search

Theorem. \( k \)-local search gives a \( \left( 1 - O\left( \frac{1}{\sqrt{k}} \right) \right) \)-approximation for …

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… Delaunay graphs are planar [M. and Ray, 2010.]

independent set

… bipartite intersection graphs are planar [Chan and Har-Peled, 2012]

terrain guarding

… visibility graph is planar [Krohn et al. 2014.]

minor-free graph families

… expansion theorem holds [Cabello and Gajser, 2015.]

dominating set

… Voronoi diagrams are planar [Govindarajan et al., 2016.]
Hall’s Theorem
Hall’s Theorem

‘Local’ version of Hall’s theorem

Let $G = (B, R, E)$ be a planar graph such that

$$\forall B' \subseteq B, \quad |B'| \leq k \quad \implies \quad |\text{Neighbors}(B')| \geq |B'|$$
Hall’s Theorem

‘Local’ version of Hall’s theorem

Let $G = (B, R, E)$ be a planar graph such that

$$\forall B' \subseteq B, \quad |B'| \leq k \quad \implies \quad |\text{Neighbors}(B')| \geq |B'|$$

**Theorem:** matching in $G$ of size at least $\left(1 - O\left(\frac{1}{\sqrt{k}}\right)\right) |B|$. 
Hall’s Theorem

‘Local’ version of Hall’s theorem

Let $G = (B, R, E)$ be a planar graph such that

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**Theorem:** matching in $G$ of size at least $\left(1 - O\left(\frac{1}{\sqrt{k}}\right)\right) |B|$.

a ‘local to global’ phenomenon in geometry
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Helly’s theorem, covering theorems, unions by inclusion-exclusion, …
Thank you