The Kuperberg conjecture for translates of convex bodies

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Introduction: packing and covering densities

- $C$ is a $d$-dimensional convex body.
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- An arrangement $\mathcal{C} = \{ C_i : i \in I \}$ is a (countable) collection of congruent copies of $C$ in $\mathbb{R}^d$. It is a packing if no two copies intersect in the interior. It is called a covering if $\mathbb{R}^d = \bigcup_{i \in I} C_i$. 

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- An arrangement is called translative if all copies are translates of $C$. It is called periodic if, in addition, the translation vectors form a set $\Lambda + X$, where $\Lambda$ is a lattice and $X$ is finite. It is called a lattice arrangement if, in addition, $|X| = 1$, i.e. the translation vectors form a lattice.
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- The upper and lower densities of an arrangement are

$$\overline{\text{den}}(\mathcal{C}) = \limsup_{r \to \infty} \sum_{i \in I} \frac{\text{vol}(C_i \cap B^d(r))}{\text{vol}(B^d(r))},$$

$$\underline{\text{den}}(\mathcal{C}) = \liminf_{r \to \infty} \sum_{i \in I} \frac{\text{vol}(C_i \cap B^d(r))}{\text{vol}(B^d(r))}.$$
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- The *packing density* of $C$ is
  \[
  \delta(C) = \sup_{\mathcal{C} \text{ is a packing}} \text{den}(\mathcal{C}).
  \]

- The *covering density* of $C$ is
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- If we restrict our attention to translatative, periodic or lattice arrangements only, then we add the respective subscript to our notation. We have
  \[
  \delta_L(C) \leq \delta_P(C) \leq \delta_T(C) \leq \delta(C) \leq 1 \leq \theta(C) \leq \theta_T(C) \leq \theta_P(C) \leq \theta_L(C).
  \]
Introduction: packing and covering densities

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- For periodic arrangements $\text{den}(C) = \frac{|X|\text{vol}(C)}{\text{vol}(\mathbb{R}^d/\Lambda)}$.

- Schmid: $\delta(C) = 1 \iff \theta(C) = 1 \iff C$ is a tile.
- The same works, if we restrict ourselves to translative or lattice densities.
- Is this property stable?

Conjecture (Kuperberg)

Let $d \geq 2$ be fixed. Then for any $\epsilon > 0$ there exists $\delta > 0$ with the property that for every $d$-dimensional convex body $C$

1. $\theta(C) \geq 1 + \epsilon$ implies $\delta(C) \leq 1 - \delta$,
2. $\delta(C) \leq 1 - \epsilon$ implies $\theta(C) \geq 1 + \delta$.

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Conjecture (Kuperberg)

Let \( d \geq 2 \) be fixed. Then for any \( \varepsilon > 0 \) there exists \( \delta > 0 \) with the property that for every \( d \)-dimensional convex body \( C \)

\[
(1) \quad \theta(C) \geq 1 + \varepsilon \quad \text{implies} \quad \delta(C) \leq 1 - \delta,
\]

\[
(2) \quad \delta(C) \leq 1 - \varepsilon \quad \text{implies} \quad \theta(C) \geq 1 + \delta.
\]
Main result

Theorem (P., 2018)

(1a) Let either $0 < \varepsilon \leq \frac{1}{d^{d+1}}$ or $C$ in addition be centrally symmetric. Then $\delta_T(C) > 1 - \varepsilon$ implies

$$\theta_T(C) < \left(1 + \varepsilon \frac{1}{d^{d+1}}\right)^{d+1}.$$

(1b) Let $\frac{1}{d^{d+1}} < \varepsilon < 1$ and $C$ be not centrally symmetric. Then $\delta_T(C) > 1 - \varepsilon$ implies

$$\theta_T(C) < \left(1 + \varepsilon d^d\right) \left(1 + \frac{1}{d}\right)^d.$$

(2) Let $0 < \varepsilon < 1$. Then $\theta_T(C) < 1 + \varepsilon$ implies

$$\delta_T(C) > \left(1 - \varepsilon \frac{1}{d^{d+1}}\right)^{d+1}.$$
Proof idea

Consider a dense periodical packing satisfying $\text{den}(C) > 1 - \varepsilon$. 
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Proof idea

There could be thin empty spaces, which is hard to cover by new copies.
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Proof idea
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Now put new translates covering thick spaces.
Proof idea

Need to control the number of new translates and the homothety ratio.
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- Project $C$ to the torus $T = \mathbb{R}^d / \Lambda$. Let $X_0 = X$ be the set of translations of $C$ on $T$. Fix $0 \leq \alpha \leq 1$. 
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- If $y \notin \bigcup_{i \in I} ((1 + \alpha)C + x_i)$, then $(-\alpha C + y) \notin \bigcup_{i \in I} (C + x_i)$. 
Proof idea

- If \((1 + \alpha) C + X_0\) is not a covering of \(T\), take a not covered point \(y\). Then \(C + y\) covers a “large” portion of \(T\) not covered by \(C + X_0\). (In non-centrally symmetric case we need \(\alpha \leq 1/d\).)
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- Let \(X_1 = X_0 \cup \{y\}\). Continue in the same way until \((1 + \alpha)C + X_l\) covers \(T\).
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- Let \(X_1 = X_0 \cup \{y\}\). Continue in the same way until \((1 + \alpha)C + X_l\) covers \(T\).
- Control the volume of the space not covered by \(C + X_k\). At each step it decreases by at least \(\alpha^d \text{vol}(C)\).
Proof idea

- If \((1 + \alpha)C + X_0\) is not a covering of \(T\), take a not covered point \(y\). Then \(C + y\) covers a “large” portion of \(T\) not covered by \(C + X_0\). (In non-centrally symmetric case we need \(\alpha \leq 1/d\).)
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- Control the volume of the space not covered by \(C + X_k\). At each step it decreases by at least \(\alpha^d \text{vol}(C)\).
- Then \(l < \varepsilon / (\alpha^d \text{vol}(C))\) and

\[
\text{den} ((1 + \alpha) C + X_l + \Lambda) \leq \left(1 + \frac{\varepsilon}{\alpha^d}\right) (1 + \alpha)^d.
\]
Proof idea of (b)

- Start from a periodic economical covering (such that the lattice is “sufficiently big”). Each step try to shrink slightly each body. If there are two bodies intersecting, delete one of them. The number of steps is controlled by the *covering excess* of the arrangement.
Proof idea of (b)

- Start from a periodic economical covering (such that the lattice is “sufficiently big”). Each step try to shrink slightly each body. If there are two bodies intersecting, delete one of them. The number of steps is controlled by the covering excess of the arrangement.

- If \( y \in ((1 - \alpha) C + x) \cap ((1 - \alpha) C + x') \), then \( (\alpha C + y) \subset ((1 - \alpha) C + x) \cap ((1 - \alpha) C + x') \)
Open questions

Question

Is the dependence on $d$ necessary? I.e. is it true that for any $\varepsilon > 0$ there exists $\mu > 0$ with the property that for every $d$ and every $d$-dimensional convex body $C$,

1. $\delta_T(C) \leq 1 - \varepsilon$ implies $\theta_T(C) \geq 1 + \mu$,
2. $\theta_T(C) \geq 1 + \varepsilon$ implies $\delta_T(C) \leq 1 - \mu$. 

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\[
\begin{align*}
(1) \quad & \delta_T(C) \leq 1 - \varepsilon \quad \text{implies} \quad \theta_T(C) \geq 1 + \mu, \\
(2) \quad & \theta_T(C) \geq 1 + \varepsilon \quad \text{implies} \quad \delta_T(C) \leq 1 - \mu.
\end{align*}
\]

Question

Is the Kuperberg conjecture true when restricted to lattice arrangements only?
The end

Thank you!

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