Fair partition of a convex planar pie

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The problem statement

**Question (Nandakumar and Ramana Rao, 2008)**

*Given a positive integer $m$ and a convex body $K$ in the plane, can we cut $K$ into $m$ convex pieces of equal areas and perimeters?*

Such partitions are called **fair**.
Previously known results

- For $m = 2$ it can be done by a simple continuity argument.

Split in two parts of equal area and rotate. At some point the perimeters must be equal.
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  ![Diagram](image)

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- A generalization of the continuity argument through an appropriate Borsuk–Ulam-type theorem yields a proof for $m = p^k$ with $p$ prime. The topological tool was used previously by Viktor Vassiliev for a different problem (1989). The fair partition result for $m = 2^k$ was proved explicitly by Mikhail Gromov (2003).
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- For $m$, which is not a prime power, this direct technique fails.
A classical example: the ham sandwich theorem

**Theorem**

Any 3 sufficiently nice probability measures in $\mathbb{R}^3$ can be simultaneously equipartitioned by a plane.

https://curiosamathematica.tumblr.com

A proof uses the well-known technique of configuration spaces, test maps, and equivariant obstructions.
Theorem (Karasev, Hubard, Aronov, Blagojević, Ziegler, 2014)

If \( m \) is a power of a prime then any convex body \( K \) in the plane can be partitioned into \( m \) parts of equal area and perimeter.

The case \( m = 3 \) was done first by Bárány, Blagojević, and Szűcs. In dimension \( n \geq 3 \) a similar result with equal volumes and equal \( n - 1 \) other continuous functions of \( m \) convex parts was also established for \( m = p^k \).
Configuration space

$F(m)$ is the space of $m$-tuples of pairwise distinct points in $\mathbb{R}^2$. Given $\bar{x} \in F(m)$ we can use Kantorovich theorem on optimal transportation to equipartition $K$ into $m$ parts of equal area. The partition is a weighted Voronoi diagram with centers in $\bar{x}$.

$\bar{x} \in F(3)$. 
Further simplification of $F(m)$

The dimension of $F(m)$ is $2m$. We can further simplify it.

**Lemma (Blagojević and Ziegler, 2014)**

Space $F(m)$ retracts $\mathfrak{S}_m$-equivariantly to its subpolyhedron $P(m) \subset F(m)$ with $\dim P(m) = m - 1$.

The set of fair partitions in $P(m)$ is discrete (dimensions counting).
Equivariant map

The test map

\[ f : P(m) \rightarrow \mathbb{R}^m \]

sends a weighted Voronoi equal area partition into the perimeters of the \(m\) parts. Map \(f\) is \(\mathcal{S}_m\)-equivariant with \(\mathcal{S}_m\) acting on \(\mathbb{R}^m\) by permutations of the coordinates. A partition corresponding to \(u \in P(m)\) is fair if

\[ f(u) \in \Delta := \{(x, x, \ldots, x) \in \mathbb{R}^m\}. \]
Homology of the solution set

**Theorem (Blagojević and Ziegler)**

If \( m = p^k \) is a prime power and \( f : P(m) \to \mathbb{R}^m \) is an \( S_m \)-equivariant map in general position, then \( f^{-1}(\Delta) \) is a non-trivial 0-dimensional cycle modulo \( p \) in homology with certain twisted coefficients.

If \( m \) is not a prime power then there exists an \( S_m \)-equivariant map \( f : P(m) \to \mathbb{R}^m \) with \( f^{-1}(\Delta) = \emptyset \).
Our main result

Theorem (Akopyan, A., Karasev)

For any $m \geq 2$ any convex body $K$ in the plane can be partitioned into $m$ parts of equal area and perimeter.
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For any $m \geq 2$ any convex body $K$ in the plane can be partitioned into $m$ parts of equal area and perimeter.

When $m$ is not a prime power, the theorem was unknown even for simplest $K$, e.g., for generic triangles.
"Naive" continuity argument

- "Naive" argument for $m = 6$ (the smallest non-prime-power):

![Diagram showing the process of picking a direction, halving, partitioning, and rotating.]
“Naive” continuity argument

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  - Pick a direction and a halving line in that direction.
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  - Pick a direction and a halving line in that direction.
  - Fair partition each half into 3 pieces.
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  - Pick a direction and a halving line in that direction.
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  - Rotate the direction.
“Naive” continuity argument

“Naive” argument for $m = 6$ (the smallest non-prime-power):

- Pick a direction and a halving line in that direction.
- Fair partition each half into 3 pieces.
- Rotate the direction.

There are difficulties arguing this way, because once we rotate by $\pi$ we might arrive to a different pair of 3-partitions.
Proof sketch for $m = 6$

As we rotate the direction, plot the perimeters of all the solutions, the language of multivalued functions might be useful.

Red and blue graphs are perimeters on the two sides of the halving line. Red/blue intersections are fair partitions.
Homology of red/blue intersection

Red and blue graphs are 1-dimensional cycles modulo 3. The homology class of their intersection is trivial.
Proof sketch for $m = 6$

Red graph separates the bottom from the top, from the homology modulo 3 description of the solution set by Blagojević and Ziegler.
Proof sketch for $m = 6$

Bold red and bold blue graphs are nontrivial 1-dimensional cycles modulo 2. The homology class of their intersection is non-trivial.
Generalizations:

- “Area” can be any finite Borel measure, zero on hyperplanes.
- “Perimeter” can be any Hausdorff-continuous function on convex bodies (e.g., diameter).
- Unknown, if we replace “area” with an arbitrary (i.e., non-additive) rigid-motion-invariant continuous function of convex bodies.
- If we want to equalize the volumes and two perimeter-like functions in $\mathbb{R}^3$, then it is possible for $m = p^k$ (K., Aronov, Hubard, Blagojević, Ziegler), but our current method does not work already for $m = 2p^k$.

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Thank you for your attention!

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