Combinatorial aspects of reduced polytopes

Thomas Jahn

July 9, 2019
Question

Given a polytope in $\mathbb{R}^d$, can we map vertices injectively to non-incident facets?

Yes for $d = 1$ and $d = 2$. What about $d \geq 3$?
**Once upon a time at DGD 2016**

**Theorem**

*Polytopes in $\mathbb{R}^3$ which have a vertex $v$ with an *antipodal* facet $F$ [...] such that the edges and facets incident to $v$ are antipodal to the edges and vertices of $F$, respectively, are not *reduced*."

A. Polyanskii  
*On reduced polytopes*,  
**Reducedness**

The **minimum width** of $P = \text{co} \{v_1, \ldots, v_n\}$ is

$$\omega(P) = \min \max_{\|u\|=1, i, j=1, \ldots, n} \langle u | v_i - v_j \rangle.$$

**Definition**

A polytope $P$ is **reduced** if $\omega(P') < \omega(P)$ for all polytopes $P' \subsetneq P$.

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E. Heil  
*Kleinste konvexe Körper gegebener Dicke*,  
Definition (Heil 1978)

A polytope $P$ is **reduced** if $\omega(P') < \omega(P)$ for all polytopes $P' \subsetneq P$. 
LASSAK–AVERKOV–MARTINI THEOREM

**Theorem**

\[ P \text{ is reduced. } \iff \text{For every vertex } v \text{ of } P, \text{ there exists an antipodal facet } F \text{ of } P \text{ at distance } \omega(P). \]

- **M. Lassak**
  *Characterizations of reduced polytopes in finite-dimensional normed spaces,*

- **G. Averkov, H. Martini**
  *Reduced polytopes and antipodality,*
CONSEQUENCES

Antipodal faces are those maximal faces touched by a pair of parallel hyperplanes.

As reduced polytopes must have non-empty interior, antipodal faces are disjoint.

Therefore, the L-A-M theorem states that for reduced polytopes, antipodality maps vertices injectively to non-incident facets.
**Polyanskii’s criterion**

**Theorem**

*Polytopes in $\mathbb{R}^3$ which have a vertex $v$ with an antipodal facet $F$ [...] such that the edges and facets incident to $v$ are antipodal to the edges and vertices of $F$, respectively, are not reduced.*

A. Polyanskii  
*On reduced polytopes,*  
How does one find reduced polytopes? I

1. Start with a near miss.
2. Modify.
3. Apply Newton’s method.
4. Profit.

B. González Merino, T. J., A. Polyanskii, G. Wachsmuth
Hunting for reduced polytopes,
HOW DOES ONE FIND REDUCED POLYTOPES? II

1. Choose a combinatorial type of a polytope. How many are there?
How does one find reduced polytopes? II

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2. Choose an injective mapping from vertices to non-incident facets. Are there any? How many?
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2 1/2. In $\mathbb{R}^3$: Forget about the ones violating Polyanskii’s criterion.

3. Find coordinates for the vertices such that this mapping is the antipodality mapping. How?

4. Adjust coordinates to find a reduced polytope. How?
# Combinatorial types of polytopes in $\mathbb{R}^3$

<table>
<thead>
<tr>
<th># facets</th>
<th># combinatorial types</th>
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</thead>
<tbody>
<tr>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
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<td>6</td>
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<td>18</td>
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</table>
COMBINATORIAL TYPES OF POLYTOPES

For reducedness, we need \( \#\text{vertices} \leq \#\text{facets} \).

dualizing
**Injective mappings from vertices to non-incident facets**

**Theorem (J./Winter 2019+)**

For $1 \leq d \leq 6$, the vertices of a polytope $P \subset \mathbb{R}^d$ with $\#\text{vertices} \leq \#\text{facets}$ can be mapped injectively to non-incident facets.

**Observation (J./Winter 2019+)**

For $d \geq 7$, there are polytopes $P \subset \mathbb{R}^d$ with $\#\text{vertices} \leq \#\text{facets}$ whose vertices cannot be mapped injectively to non-incident facets.

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T. J., M. Winter

*Vertex-facet assignments for polytopes*,

arxiv:1812.08640
Theorem ensures the existence of injective mappings from vertices to non-incident facets for all polytopes in $\mathbb{R}^3$.

Let us enumerate them.
**It’s the small things that count.**

Results for polytopes in $\mathbb{R}^3$ with ...

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>... 6 vertices or facets:</td>
<td>1083</td>
<td>5</td>
</tr>
<tr>
<td>... 7 vertices or facets:</td>
<td>108712</td>
<td>504</td>
</tr>
<tr>
<td>... 8 vertices or facets:</td>
<td>16177054</td>
<td>53709</td>
</tr>
</tbody>
</table>

$A =$ injective mappings from vertices to non-incident facets...  
$B =$ ... which violate “Polyanskii’s criterion”
Thanks for your attention!