Around Radon’s number

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Theorem (Radon, 1921)

Any set of $d + 2$ points in $\mathbb{R}^d$ can be partitioned into two disjoint sets whose convex hulls intersect.
Convex hull relative to a set system

- $\mathcal{F} = \{F_1, \ldots, F_n\}$ subsets of a ground set $X$
- $S \subseteq X$, $\text{conv}_\mathcal{F}(S) = \bigcap\{F_i \in \mathcal{F} : S \subseteq F_i\}$
- If $S \not\subseteq F_i$ for any $i$, $\text{conv}_\mathcal{F}(S) = X$
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\[
\begin{align*}
F_1 & \subseteq X \\
F_2 & \subseteq X \\
F_3 & \subseteq X \\
S & \subseteq X \\
X & = \mathbb{R}^d
\end{align*}
\]

\( \mathcal{F} = \) convex sets
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\( X = \mathbb{R}^d \)

\( \mathcal{F} = \) convex sets
Radon’s number $r(\mathcal{F})$ of a family $\mathcal{F}$
the smallest $r$ s.t. any set $S \subseteq X$, $|S| = r$, can be split into two parts $S = P_1 \sqcup P_2$ satisfying $\text{conv}_{\mathcal{F}}(P_1) \cap \text{conv}_{\mathcal{F}}(P_2) \neq \emptyset$. 
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- $r(\text{convex sets in } \mathbb{R}^d) \leq d + 2$
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Radon’s number provides bounds for many other parameters of $\mathcal{F}$ (more on that later)
Topological complexity

- \( \mathcal{F} \) set system in a topological space of dimension \( d \)
- \( k \in \mathbb{Z}_+ \cup \{ \infty \} \)

\( k \)-level topological complexity of \( \mathcal{F} \):

\[
TC_k(\mathcal{F}) = \sup\{ \tilde{\beta}_i(\bigcap G): G \subseteq \mathcal{F}, 0 \leq i < k \},
\]

where \( \tilde{\beta}_i \) – reduced Betti numbers with \( \mathbb{Z}_2 \)-coefficients
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Families $\mathcal{F}$ with bounded topological complexity $TC_\infty(\mathcal{F})$:

- convex sets in $\mathbb{R}^d$
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Families $\mathcal{F}$ with bounded topological complexity $TC_{\infty}(\mathcal{F})$:

- convex sets in $\mathbb{R}^d$
- good covers
- hollow boxes
- spheres and pseudospheres
- finite families of semialgebraic sets in $\mathbb{R}^d$ with bounded description complexity...
“Bounded top. complexity ⇒ bounded Radon’s number”

Theorem

For $b, d \geq 0$ there is a number $r(b, d)$ s.t. the following holds: If $\mathcal{F}$ is a finite family of sets in $\mathbb{R}^d$ with $TC_{\lceil d/2 \rceil}(\mathcal{F}) \leq b$, then $r(\mathcal{F}) \leq r(b, d)$. 
Tverberg’s number $r_k(\mathcal{F})$
the smallest $r_k$, $k \geq 3$, s.t. any set $S \subseteq X$, $|S| = r_k$, can be split into $k$ parts $P_i$ with $\bigcap_{i=1}^{k} \text{conv}_{\mathcal{F}}(P_i) \neq \emptyset$.
$r_k(\mathcal{F}) = \infty$ if there is no such $r_k$.

- $r_k(\mathcal{F}) \leq r(\mathcal{F})^{\lceil \log_2 k \rceil}$  

Jamison-Waldner ’76
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- $r_k(\mathcal{F}) \leq r(\mathcal{F})^\lceil \log_2 k \rceil$  .................. Jamison-Waldner '76

Helly’s number $h(\mathcal{F})$
the smallest $h$ s.t. if in a finite subfamily $S \subseteq \mathcal{F}$ each $h$ members of $S$ have a point in common $\Rightarrow \bigcap S \neq \emptyset$.
$h(\mathcal{F}) = \infty$ if no such $h$ exists.

- $h(\mathcal{F}) + 1 \leq r(\mathcal{F})$  .................. Levi '51
Direct consequences – Part II

Bounded $\mathcal{TC}_{\lceil d/2 \rceil}(\mathcal{F}) \Rightarrow$ bounded Radon’s number

$\Rightarrow$ bounded Helly number [Goaoc, Paták, P, Tancer, Wagner ’15]
Bounded $\sum_{d/2}^{\leq d/2} \Rightarrow$ bounded Radon’s number

$\Rightarrow$ bounded Helly number [Goaoc, Patáček, P, Tancer, Wagner ’15]

Holmsen, Lee ’19:
bounded Radon’s number $\Rightarrow$ bounded fractional Helly’s number
$\Rightarrow$ bounded colorful Helly’s number
Carathéodory’s number $c(F)$
the smallest $c$ s.t.: For any set $S$ and any point $x \in \text{conv}_F(S)$, there is a subset $S' \subseteq S$, $|S'| \leq c$, and $x \in \text{conv}_F(S')$.
$c(F) = \infty$ if no such $c$ exists.
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Bounded top. complexity $\nRightarrow$ bounded Car. number
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Bounded top. complexity $\nRightarrow$ bounded Car. number

Theorem
For any $c \geq 2$, $d \geq 2$ there is a finite family $\mathcal{F}$ of sets in $\mathbb{R}^d$ with $TC_\infty(\mathcal{F}) = 0$ and $c(\mathcal{F}) = c$. 

![Diagram showing a set of points and subsets](image)
fractional Helly in the plane

$m_i(G) =$ number of intersecting $i$-tuples in $G$

**Theorem**

Fix $b \geq 0$. $\forall \alpha \in (0, 1) \quad \exists \beta = \beta(\alpha, b) > 0$ s.t.:

- If $F$ is a family of open sets in $\mathbb{R}^2$ with $TC_1(F) \leq b$, and
- $G$ is a finite subfamily of $F$-convex sets with $m_3(G) \geq \alpha(|G|)$

$\Rightarrow$ there is a point in common to at least $\beta|G|$ sets of $G$.

- $S$ is $F$-convex if $\text{conv}_F(S) = S$
- holds also for compact connected surfaces
A variant of fractional Helly in the plane

\( m_i \) = number of intersecting \( i \)-tuples

**Theorem (Kalai, P ’19)**

Let \( b \geq 0, \ell \geq 3 \) and let \( F \subseteq \mathbb{R}^2 \) be a family of open sets s.t.

- \( m_{\ell+1} = 0 \)
- intersection of any \((\ell - 1)\)-tuple or any \( \ell \)-tuple of sets has \( \leq (b + 1) \) path-connected components

\( \Rightarrow m_\ell \leq cm_{\ell-1} \) for some \( c = c(b, \ell) \).

In particular, \( m_\ell \leq c\binom{n}{\ell-1} \).

**Note:** holds also for compact connected surfaces
Lemma

- \( \ell \geq 4, b \geq 0 \) fixed parameters
- \( F \) a family of \( n \) open sets in \( \mathbb{R}^2 \) with \( TC_1(F) \leq b \)
- then \( \forall \alpha \in (0, 1) \; \exists \beta = \beta(\alpha, b, \ell) > 0 \) such that

\[
m_3 \geq \alpha \left( \frac{n}{3} \right) \Rightarrow m_\ell \geq \beta \left( \frac{n}{\ell} \right)
\]
Lemma

• $\ell \geq 4$, $b \geq 0$ fixed parameters
• $\mathcal{F}$ a family of $n$ open sets in $\mathbb{R}^2$ with $TC_1(\mathcal{F}) \leq b$
• then $\forall \alpha \in (0, 1)$ $\exists \beta = \beta(\alpha, b, \ell) > 0$ such that

\[ m_3 \geq \alpha \left( \frac{n}{3} \right) \Rightarrow m_\ell \geq \beta \left( \frac{n}{\ell} \right) \]

Remarks:

• combined with the result of Holmsen and Lee

$\Rightarrow$ fract. Helly number is 3 in $\mathbb{R}^2$

• Lemma & fract. Helly hold also for connected surfaces
\( \mathcal{F} \) has \((p, q)\)-property if among every \( p \) sets of \( \mathcal{F} \), some \( q \) have a point in common

**Theorem**

\( \forall p \geq q \geq 3, b \geq 0 \) and a surface \( S \), \( \exists C = C(p, q, \chi(S)) \) s.t.:

- \( \mathcal{F} \) a finite family of open subsets of \( S \) with \( TC_1(\mathcal{F}) \leq b \)

\( \mathcal{F} \) has the \((p, q)\)-property \( \Rightarrow \) \( \mathcal{F} \) can be pierced by \( \leq C \) elements.
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**Theorem**

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**Note:** \( b = 0 \) settles a conjecture by Holmsen, Kim, and Lee!
Bounded $TC_{\lceil d/2 \rceil}(\mathcal{F}) \Rightarrow$ bounded Radon’s number

**Radon’s number $r(\mathcal{F})$ (reminder)**
is the smallest $r$ s.t. $\forall S \subseteq X, |S| = r, \exists P_1, P_2 \subseteq S: S = P_1 \sqcup P_2$ and $\text{conv}_\mathcal{F}(P_1) \cap \text{conv}_\mathcal{F}(P_2) \neq \emptyset$.

**Proof idea for a point set $S \subseteq \mathbb{R}^2$:**
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We want Two disjoint subsets $P_1, P_2 \subseteq S$ whose convex hulls intersect
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**Proof idea for a point set $S \subseteq \mathbb{R}^2$:**

- **We want** Two disjoint subsets $P_1, P_2 \subseteq S$ whose convex hulls intersect
- **We know** In every drawing of $K_5$ two disjoint edges intersect (Hanani-Tutte)
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- **Idea** Draw each point/edge of $K_5$ inside $\text{conv}_\mathcal{F}(Q)$ for a suitable set $Q$
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We know In every drawing of $K_5$ two disjoint edges intersect (Hanani-Tutte)

Idea Draw each point/edge of $K_5$ inside $\text{conv}_{\mathcal{F}}(Q)$ for a suitable set $Q$

Require Disjoint edges $\Rightarrow$ disjoint sets $Q$
Definition (Constraint map)
A drawing $\varphi$ of $K_5$ into $\mathbb{R}^2$ is constraint by $(S, F)$, if there is $\Psi : E(K_5) \cup V(K_5) \cup \{\emptyset\} \rightarrow 2^S$ satisfying

- $\Psi(\emptyset) = \emptyset$
- $\Psi(\sigma \cap \tau) = \Psi(\sigma) \cap \Psi(\tau)$
- $\varphi(x) \subseteq \text{conv}_F \Psi(x)$ for each $x \in E(K_5) \cup V(K_5)$. 
Path-connected intersections in $\mathbb{R}^2$ ($b = 0$, $d = 2$)

- Let $S = \{p_1, p_2, \ldots, p_5\}$
- Map $i$th vertex $v_i$ of $K_5$ to $p_i$ and set $\Psi(v_i) := \{p_i\}$
- $b = 0$, so for every $i \neq j$, $\text{conv}_{\mathbb{F}}\{p_i, p_j\}$ is connected
Path-connected intersections in $\mathbb{R}^2 (b = 0, d = 2)$

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- $b = 0$, so for every $i \neq j$, $\text{conv} \mathcal{F}\{p_i, p_j\}$ is connected
  $\Rightarrow$ it suffices to set $\Psi(v_i, v_j) := \{p_i, p_j\}$

We have just proved that $r(F) \leq 5$, which is sharp.
Path-connected intersections in $\mathbb{R}^2$ ($b = 0$, $d = 2$)

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We have just proved that $r(F) \leq 5$, which is sharp.
Case $b = 1$:

- among any three distinct points $p_i, p_j, p_k$, some two can be connected inside $\text{conv}_F\{p_i, p_j, p_k\}$

- color each triple by the “position” of the connected pair
Case $b = 1$:

- among any **three distinct** points $p_i, p_j, p_k$, **some two** can be connected inside $\text{conv}_F\{p_i, p_j, p_k\}$
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**Ramsey:** for $|S| \geq R_3(15; 3) \exists$ a monochr. set $Z$ of 15 pts
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**Ramsey:** for $|S| \geq R_3(15; 3)$ $\exists$ a monochr. set $Z$ of 15 pts

- find $\Psi : \{\emptyset\} \cup V(K_5) \cup E(K_5) \rightarrow 2^Z$ that
  - vertex $v_i \mapsto$ point $p_i \in Z$
  - edge $v_i v_j \mapsto$ 3-element set $\Psi(v_i v_j)$ s.t. $p_i, p_j$ are connected inside $\text{conv}_F\Psi(v_i v_j)$ and $\Psi(v_i v_j)$ are disjoint for disjoint edges.
Raising $b$ in $\mathbb{R}^2$

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- conclude $r(F) \leq R_3(15; 3)$
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  - vertex $v_i \mapsto$ point $p_i \in Z$
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    inside $\text{conv}_\mathcal{F} \Psi(v_i v_j)$ and $\Psi(v_i v_j)$ are **disjoint** for disjoint edges.

- conclude $r(\mathcal{F}) \leq R_3(15; 3)$

For $b \geq 2$, $r(\mathcal{F}) \leq R_{b+2} \left(5 + 10b; \left\lceil \frac{b+2}{2} \right\rceil \right)$. 
For $d \geq 3$ we proceed as follows:

- induction
- being in the same component $\leadsto$ (subdivided) cycle with trivial homology
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- maps $\rightsquigarrow$ chain maps
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- $K_5 \rightsquigarrow \lceil d/2 \rceil$-skeleton of $(d+2)$-simplex

[chain map generalization of Van-Kampen Flores thm]
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- induction
- being in the same component $\leadsto$ (subdivided) cycle with trivial homology
- maps $\leadsto$ chain maps
- $K_5 \leadsto \lceil d/2 \rceil$-skeleton of $(d + 2)$-simplex

[chain map generalization of Van-Kampen Flores thm]

Thanks for your time!